TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 274, Number 2, December 1982

EISENSTEIN SERIES OF WEIGHT 3/2. I

BY

TING-YI PEI

ABSTRACT. We prove that in the space of elliptic modular forms with weight 3/2, the orthogonal complement of the subspace of cusp forms with respect to the Petersson inner product is generated by Eisenstein series in some special cases.

1. Introduction. It is a well-known fact that in the space of elliptic modular forms, the orthogonal complement of the subspace of cusp forms with respect to the Petersson inner product is generated by Eisenstein series if the weight is an integer or a half integer $\geqslant 5/2$ (see Hecke [**2**] and Petersson [**4**]). The result of Serre and Stark in [**5**] gives a somewhat different but similar decomposition for weight 1/2. In this paper we discuss the same problem for weight 3/2. We shall prove that under certain conditions on the level, the orthogonal complement is indeed generated by some Eisenstein series which are explicitly constructed. We plan to discuss more general cases in a subsequent paper.

To state our result more precisely, let us introduce the notion of modular form of half integral weight, following Shimura [**6**]. Put

$$\theta(z) = \sum_{n=-\infty}^{+\infty} e(n^2 z),$$

where z is a variable on the upper half plane

$$H = \{z = x + iy \in \mathbf{C} \mid y > 0\}$$

and $e(z) = e^{2\pi i z}$. It is known that

$$\theta(\gamma(z)) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}(cz+d)^{1/2}\theta(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, where $\gamma(z) = (az+b)(cz+d)^{-1}$ and

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv 3 \pmod 4, \end{cases}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \pmod N \right\}.$$

The symbol $(\frac{c}{d})$ means the quadratic residue. For $z \in \mathbf{C}$, we define $z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leqslant \pi/2$. Put

$$j(\gamma, z) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}(cz+d)^{1/2} \quad \text{for } \gamma \in \Gamma_0(4). \tag{1.1}$$

Received by the editors September 9, 1981.
1980 *Mathematics Subject Classification*. Primary 10D12.

0002-9947/81/0000-0270/$08.50

We have, if $\gamma_1, \gamma_2 \in \Gamma_0(4)$,

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2(z))j(\gamma_2, z). \tag{1.2}$$

Throughout the paper we let κ denote an odd positive integer, N a positive integer divisible by 4, $N = \prod p^{e(p)}$ its prime decomposition, and ω a Dirichlet character modulo N with $\omega(-1) = 1$. A holomorphic function $f(z)$ on H is called a *modular form of weight $\kappa/2$ and character ω* if

(i) $f(\gamma(z)) = \omega(d)j(\gamma, z)^{\kappa}f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,

(ii) f is holomorphic at the cusps of $SL_2(\mathbf{Z})$.

The second condition means that for any cusp d/c, there exists an element $\rho = \left(\begin{smallmatrix} a & b \\ -c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ such that $\rho(d/c) = i\infty$ and $f(\rho^{-1}(z))(cz + a)^{-\kappa/2}$ is holomorphic at $z = i\infty$. The complex vector space of all such f is denoted by $M(N, \kappa/2, \omega)$. We then denote by $S(N, \kappa/2, \omega)$ the subspace of cusp forms in $M(N, \kappa/2, \omega)$, and by $\mathfrak{S}(N, \kappa/2, \omega)$ the orthogonal complement of $S(N, \kappa/2, \omega)$ in $M(N, \kappa/2, \omega)$. We write $M(N, \omega)$, $S(N, \omega)$ and $\mathfrak{S}(N, \omega)$ instead of $M(N, 3/2, \omega)$, $S(N, 3/2, \omega)$ and $\mathfrak{S}(N, 3/2, \omega)$ respectively in this paper. We consider also the group extension G of $GL_2^+(\mathbf{R})$ which consists of all the pairs $\{\gamma, \phi(z)\}$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbf{R})$ and $\phi(z)^2 = \alpha \det(\gamma)^{-1/2}(cz + d)$ with $|\alpha| = 1$. The multiplication law in G is given by

$$\{\gamma_1, \phi_1(z)\} \cdot \{\gamma_2, \phi_2(z)\} = \{\gamma_1\gamma_2, \phi_1(\gamma_2(z))\phi_2(z)\} \qquad (\gamma_1, \gamma_2 \in GL_2^+(\mathbf{R})).$$

If $\xi = \{\gamma, \phi(z)\} \in G, f \in M(N, \omega)$, we define the operator $f \mid \xi$ by

$$(f \mid \xi)(z) = \phi(z)^{-3}f(\gamma(z)).$$

Put $\Gamma_\infty = \{\pm\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \mid m \in \mathbf{Z}\}$. Let us define functions $E(s, \omega, N)$ and $E'(s, \omega, N)$ by

$$E(s, \omega, N)(z) = y^{s/2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \omega(d_\gamma) j(\gamma, z)^{-3} |j(\gamma, z)|^{-2s},$$

$$E'(s, \omega, N)(z) = E(s, \omega, N)(-1/Nz)z^{-3/2},$$

where d_γ is the lower right entry of γ, $z \in H$ and $s \in \mathbf{C}$, $\mathrm{Re}(s) > \frac{1}{2}$. Let χ_d and χ'_d denote the primitive characters such that

$$\chi_d(k) = \left(\frac{d}{k}\right) \quad \text{and} \quad \chi'_d(k) = \left(\frac{k}{d}\right) \quad \text{for } (k, 4d) = 1.$$

Now, if $\omega^2 \neq \mathrm{id}$, we define an Eisenstein series $E(\omega, N)$ by

$$E(\omega, N)(z) = E(0, \bar{\omega}, N)(z).$$

If $N = 4D$ or $8D$, where D is an odd square-free integer, we define Eisenstein series $f_1(\mathrm{id}, 4D)$ and $f_1(\mathrm{id}, 8D)$ by

$$f_1(\mathrm{id}, 4D) = E(0, \mathrm{id}, 4D) - (1 - i)(4D)^{-1}E'(0, \chi_D, 4D).$$

$$f_1(\mathrm{id}, 8D) = E(0, \mathrm{id}, 8D) - (1 - i)(8D)^{-1}E'(0, \chi_{2D}, 8D).$$

We shall show (Lemma 2.3) that $E(\omega, N)$ belongs to $\mathfrak{S}(N, \omega)$, $f_1(\mathrm{id}, 4D)$ belongs to $\mathfrak{S}(4D, \mathrm{id})$, and $f_1(\mathrm{id}, 8D)$ belongs to $\mathfrak{S}(8D, \mathrm{id})$. Our main result in this paper is

THEOREM. *Let D be an odd square-free integer, l a positive divisor of D, and e an integer $\geqslant 4$. Then the spaces $\mathfrak{S}(4D, \chi_l)$, $\mathfrak{S}(8D, \chi_l)$, $\mathfrak{S}(8D, \chi_{2l})$ and $\mathfrak{S}(2^e, \mathrm{id})$ are generated by the Eisenstein series $E(\omega, N)$ ($\omega^2 \neq \mathrm{id}$), $f_1(\mathrm{id}, 4D)$, $f_1(\mathrm{id}, 8D)$, and their transforms under suitably chosen elements of G.*

This paper was written during my stay at Princeton University. It is a pleasure to thank Professor Goro Shimura for his valuable suggestions and encouragement.

2. Eisenstein series of weight 3/2. We choose a set W of representatives for $\Gamma_\infty \setminus \Gamma_0(N)$ so that

$$(*) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \{c, d\}$$

is a one-to-one map of W onto the set W_0 of all ordered couples of integers $\{c, d\}$ such that $(c, d) = 1$, $c \equiv 0 \pmod N$ and $c \geqslant 0$; if $c = 0$, then $d = 1$. We can also choose another set W' for $\Gamma_\infty \setminus \Gamma_0(N)$ so that $(*)$ gives one-to-one map of W' onto the set W_0' of all ordered couples of integers $\{c, d\}$ such that $(c, d) = 1$, $c \equiv 0$ (mod N) and $d > 0$. Using these sets of representatives, we find that

(2.1)

$$E(s, \omega, N)(z) = y^{s/2}\left\{1 + \sum_{d=-\infty}^{\infty} \sum_{c=1}^{\infty} \omega(d)\varepsilon_d^{-1}\left(\frac{cN}{d}\right)(cNz + d)^{-3/2}|cNz + d|^{-s}\right\}$$

$$= y^{s/2} \sum_{d=1}^{\infty} \sum_{c=-\infty}^{\infty} \omega(d)\varepsilon_d^{-1}\left(\frac{cN}{d}\right)(cNz + d)^{-3/2}|cNz + d|^{-s},$$

(2.2)

$$E'(s, \omega, N)(z) = y^{s/2}N^{-s/2} \sum_{d=1}^{\infty} \sum_{c=-\infty}^{\infty} \omega(d)\varepsilon_d^{-1}\left(\frac{-cN}{d}\right)(dz + c)^{-3/2}|dz + c|^{-s}.$$

Let $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right)$ be an element of $\Gamma_0(N)$. From (1.2) we obtain

$$(2.3) \qquad E(s, \omega, N)(\gamma(z)) = \bar{\omega}(d)j(\gamma, z)^3 E(s, \omega, N)(z).$$

From Proposition 1.4 of [6] we obtain

$$(2.4) \qquad E'(s, \omega, N)(\gamma(z)) = \omega(d)\left(\frac{N}{d}\right)j(\gamma, z)^3 E'(s, \omega, N)(z).$$

Both $E(s, \omega, N)$ and $E'(s, \omega, N)$ can be analytically continued as meromorphic functions to the whole s-plane. Their Fourier expansions can be found in Shimura [7] and Sturm [9]. We state the result without proof here.

For $y > 0$, $\alpha, \beta \in \mathbf{C}$ with $\mathrm{Re}(\beta) > 0$, define

$$W(y, \alpha, \beta) = \Gamma(\beta)^{-1}\int_0^\infty (u + 1)^{\alpha-1}u^{\beta-1}e^{-yu}\,du.$$

Then $W(y, \alpha, \beta)$ can be continued to a holomorphic function in (α, β) on the whole $\mathbf{C}^2$. We shall use the facts $W(y, \alpha, 0) = 1$ and $W(y, 1, -1/2) = y^{1/2}$ later on.

Define $t_n(y, \alpha, \beta)$ for $0 < y \in \mathbf{R}$ and $n \in \mathbf{Z}$ by

(2.5)

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}t_n(y, \alpha, \beta) = \begin{cases} n^{\alpha+\beta-1}e^{-2\pi ny}\Gamma(\alpha)^{-1}W(4\pi ny, \alpha, \beta) & (n > 0), \\ |n|^{\alpha+\beta-1}e^{-2\pi|n|y}\Gamma(\beta)^{-1}W(4\pi|n|y, \beta, \alpha) & (n < 0), \\ \Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & (n = 0). \end{cases}$$

Then we have

(2.6)

$$E(s, \omega, N)(z) = y^{s/2}\left\{1 + \sum_{n=-\infty}^{\infty} a(n, s, \omega, N) t_n(y, (s+3)/2, s/2) e(nx)\right\},$$

where

$$a(n, s, \omega, N) = \sum_{c=1}^{\infty} (cN)^{-s-3/2} \sum_{d=1}^{cN} \omega(d) \varepsilon_d^{-1} \left(\frac{cN}{d}\right) e\left(\frac{nd}{cN}\right)$$

and

(2.7)

$$E'(s, \omega, N)(z) = y^{s/2} N^{-s/2} \sum_{n=-\infty}^{\infty} b(n, s, \omega, N) t_n(y, (s+3)/2, s/2) e(nx),$$

where

$$b(n, s, \omega, N) = \sum_{d=1}^{\infty} \left(\frac{-N}{d}\right) \varepsilon_d^{-1} \omega(d) d^{-s-3/2} \sum_{m=1}^{d} \left(\frac{m}{d}\right) e\left(\frac{nm}{d}\right).$$

Define primitive characters $\omega^{(n)}$ and ω' by

$$\omega^{(n)}(k) = \left(\frac{-nN}{k}\right) \omega(k), \qquad \omega'(k) = \omega(k)^2$$

for $(k, nN) = 1$. Then we can show, for $n \neq 0$, that

(2.8) $$L_N(2s+2, \omega') b(n, s, \omega, N) = L_N(s+1, \omega^{(n)}) \beta(n, s, \omega, N),$$

where

(2.9) $$\beta(n, s, \omega, N) = \sum \mu(a) \omega^{(n)}(a) \omega'(b) a^{-1-s} b^{-1-2s}.$$

The last sum is extended over all positive integers a and b prime to N such that $(ab)^2$ divides n, and μ denotes the Moebius function. For $n = 0$ we can show that

(2.10) $$b(0, s, \omega, N) = L_N(2s+1, \omega') / L_N(2s+2, \omega').$$

Here we use the notation $L_N(s, \omega) = \Sigma_{(n,N)=1} \omega(n) n^{-s}$. Further we have

(2.11) $$a(n, s, \omega, N) = b(n, s, \omega\chi_N, N) c(n, s, \omega, N),$$

where

(2.12) $$c(n, s, \omega, N) = \sum_{N|M|N^{\infty}} \sum_{d=1}^{M} \left(\frac{M}{d}\right) \omega(d) \varepsilon_d^{-1} e\left(\frac{nd}{M}\right) M^{-s-3/2}.$$

$c(n, s, \omega, N)$ is a finite Dirichlet series when $n \neq 0$.

Suppose $\omega^2 \neq \mathrm{id}$ ("id" denotes the principal character); then $s = 0$ is not a pole of $L_N(s+1, \omega^{(n)})$ for any n. Let us now consider the values of $E(s, \bar{\omega}, N)$ and $E'(s, \bar{\omega}, N)$ at $s = 0$, which will be denoted by $E(\omega, N)$ and $E'(\omega, N)$. Since

$t_n(y, 3/2, 0) = 0$ when $n \leqslant 0$ and $|c(0,0,\omega,N)| \leqslant \Sigma_{N|M|N^\infty} M^{-1/2} < +\infty$, we obtain

(2.13)

$$E(\omega, N)(z) = E(0, \bar{\omega}, N)(z)$$
$$= 1 - 4\pi(1+i)\sum_{n=1}^{\infty} \frac{L_N\big(1, (\bar{\omega}\chi_N)^{(n)}\big)}{L_N(2, \bar{\omega}')} \beta(n, 0, \bar{\omega}\chi_N, N) c(n, 0, \bar{\omega}, N) n^{1/2} e(nz),$$

(2.14) $$E'(\omega, N)(z) = E'(0, \bar{\omega}, N)(z)$$
$$= -4\pi(1+i)\sum_{n=1}^{\infty} \frac{L_N(1, \bar{\omega}^{(n)})}{L_N(2, \bar{\omega}')} \beta(n, 0, \bar{\omega}, N) n^{1/2} e(nz).$$

We shall show in Lemma 2.3 that $E(\omega, N)$ and $E'(\omega, N)$ are elements of $\mathfrak{S}(N, \omega)$ and $\mathfrak{S}(N, \bar{\omega}\chi_N)$ respectively.

In order to find more forms we need

LEMMA 2.1. *Put*

$$a(2^\nu, n) = \sum_{d=1}^{2^\nu} \left(\frac{2^\nu}{d}\right) \varepsilon_d^{-1} e\left(\frac{nd}{2^\nu}\right),$$
$$a(p^\nu, n) = \varepsilon_{p^\nu} \sum_{d=1}^{p^\nu} \left(\frac{d}{p^\nu}\right) e\left(\frac{nd}{p^\nu}\right) \qquad (p \neq 2)$$

where ν is a positive integer. Then

$$c(n, s, \mathrm{id}, N) = \prod_{p|N} \sum_{\nu=e(p)}^{\infty} p^{-(s+3/2)\nu} a(p^\nu, n).$$

PROOF. Let $M = 2^e M_1$, $e \geqslant 2$, $2 \nmid M_1$, then

$$\sum_{d=1}^{M} \left(\frac{M}{d}\right) \varepsilon_d^{-1} e\left(\frac{nd}{M}\right) = \sum_{d_1=1}^{M_1} \sum_{d_2=1}^{2^e} \left(\frac{2^e M_1}{2^e d_1 + M_1 d_2}\right) \varepsilon_{M_1 d_2}^{-1} e\left(n\left(\frac{d_1}{M_1} + \frac{d_2}{2^e}\right)\right)$$
$$= \sum_{d_2=1}^{2^e} \left(\frac{2^e}{M_1 d_2}\right) \varepsilon_{M_1 d_2}^{-1} e\left(\frac{nd_2}{2^e}\right) \cdot \sum_{d_1=1}^{M_1} \left(\frac{2^e d_1}{M_1}\right) \varepsilon_{d_2}^{-1} \varepsilon_{M_1} \varepsilon_{M_1 d_2} e\left(\frac{nd_1}{M_1}\right)$$
$$= a(2^e, n) \cdot \varepsilon_{M_1} \sum_{d_1=1}^{M_1} \left(\frac{d_1}{M_1}\right) e\left(\frac{nd_1}{M_1}\right).$$

Here we use the quadratic reciprocity

$$\left(\frac{c}{d}\right) \cdot \left(\frac{d}{c}\right) = (-1)^{(c-1)(d-1)/4} = \varepsilon_{cd}^{-1} \varepsilon_c \varepsilon_d$$

if $2 \nmid cd$ and $c > 0$, $d > 0$. Furthermore let $M_1 = M_2 M_3$ and $(M_2, M_3) = 1$, then

$$\varepsilon_{M_1} \sum_{d_1=1}^{M_1} \left(\frac{d_1}{M_1}\right) e\left(\frac{nd_1}{M_1}\right) = \varepsilon_{M_1} \sum_{d_2=1}^{M_2} \sum_{d_3=1}^{M_3} \left(\frac{d_2 M_3}{M_2}\right)\left(\frac{d_3 M_2}{M_3}\right) e\left(n\left(\frac{d_2}{M_2} + \frac{d_3}{M_3}\right)\right)$$
$$= \varepsilon_{M_2} \sum_{d_2=1}^{M_2} \left(\frac{d_2}{M_2}\right) e\left(\frac{nd_2}{M_2}\right) \cdot \varepsilon_{M_3} \sum_{d_3=1}^{M_3} \left(\frac{d_3}{M_3}\right) e\left(\frac{nd_3}{M_3}\right).$$

This proves the lemma.

By a straighforward computation we have (cf. Maass [**3**, p. 140])

$$(2.15)\quad a(2^\nu, n) = \begin{cases} 0, & 2^{\nu-2} \nmid n,\ 2 \mid \nu, \\ -2^{\nu-3/2} e^{\pi i(l/2+3/4)}, & 2^{\nu-2} \mid n,\ l = n/2^{\nu-2},\ 2 \mid l,\ 2 \mid \nu, \\ 2^{\nu-3/2} e^{\pi i(l/2+1/4)}, & 2^{\nu-2} \mid n,\ l = n/2^{\nu-2},\ 2 \nmid l,\ 2 \mid \nu, \\ 2^{\nu-1}\delta\left(\dfrac{u-3}{4}\right) e^{\pi i u/4}, & n = 2^{\nu-3}u,\ 2 \nmid u,\ 2 \nmid \nu, \\ 0 & \text{otherwise}, \quad 2 \nmid \nu, \end{cases}$$

where

$$\delta(x) = \begin{cases} 1, & x \text{ is integer}, \\ 0, & \text{otherwise}. \end{cases}$$

For $p \neq 2$ we have

$$(2.16)\qquad a(p^\nu, n) = \begin{cases} 0, & p^{\nu-1} \nmid n, \\ p^{\nu-1/2}\left(\dfrac{-n/p^{\nu-1}}{p}\right), & p^{\nu-1} \mid n,\ p^\nu \nmid n,\ 2 \nmid \nu, \\ 0, & p^\nu \mid n,\ 2 \nmid \nu, \\ -p^{\nu-1}, & p^{\nu-1} \mid n,\ p^\nu \nmid n,\ 2 \mid \nu, \\ \phi(p^\nu), & p^\nu \mid n,\ 2 \mid \nu. \end{cases}$$

Here ϕ is the Euler function.

Put

$$A(2, n) = \sum_{\nu=2}^{\infty} 2^{-3\nu/2} a(2^\nu, n),$$

$$A(p, n) = \sum_{\nu=1}^{\infty} p^{-3\nu/2} a(p^\nu, n) \qquad (p \neq 2).$$

Let $n \neq 0$ and $h(p)$ be the highest exponent such that $p^{h(p)} \mid n$. Then we have, from (2.15) and (2.16),

(2.17)

$$A(2, n) = \begin{cases} 4^{-1}(1-i)(1 - 3 \cdot 2^{-(1+h(2))/2}), & 2 \nmid h(2), \\ 4^{-1}(1-i)(1 - 3 \cdot 2^{-(1+h(2)/2)}), & 2 \mid h(2),\ n/2^{h(2)} \equiv 1 \pmod 4, \\ 4^{-1}(1-i)(1 - 2^{-h(2)/2}), & 2 \mid h(2),\ n/2^{h(2)} \equiv 3 \pmod 8, \\ 4^{-1}(1-i), & 2 \mid h(2),\ n/2^{h(2)} \equiv 7 \pmod 8, \end{cases}$$

$$(2.18)\quad A(p, n) = \begin{cases} p^{-1} - (1+p)p^{-(3+h(p))/2}, & 2 \nmid h(p), \\ p^{-1} - 2p^{-1-h(p)/2}, & 2 \mid h(p),\ \left(\dfrac{-n/p^{h(p)}}{p}\right) = -1, \\ p^{-1}, & 2 \mid h(p),\ \left(\dfrac{-n/p^{h(p)}}{p}\right) = 1, \end{cases}$$

where $p \neq 2$. If $n = 0$ we have, from (2.15) and (2.16),

$$A(2,0) = 4^{-1}(1-i), \qquad A(p,0) = p^{-1}. \tag{2.19}$$

It is easy to verify from (2.17), (2.18) and (2.19) that

$$\begin{aligned} &A(2,4n) - 4^{-1}(1-i) = 2^{-1}(A(2,n) - 4^{-1}(1-i)), \\ &A(p,p^2n) - p^{-1} = p^{-1}(A(p,n) - p^{-1}) \qquad (p \neq 2). \end{aligned} \tag{2.20}$$

Suppose $N = 4D$ where D is an odd square-free positive integer. Consider the terms with $n \leqslant 0$ in both series $E(0, \mathrm{id}, 4D)$ and $E'(0, \chi_D, 4D)$. If $-n$ is not a square, then the term with $e(nx)$ does not appear in both series, because $t_n(y, 3/2, 0) = 0$ and $L_{4D}(1, \chi_{-n})$ is finite. If $n = -m^2$ with a nonnegative integer m, $L_{4D}(1, \chi_{m^2})t_n(y, 3/2, 0)$ is finite since

$$\zeta(1+s)\Gamma^{-1}(s/2) = 2^{-1}s\zeta(1+s)\Gamma^{-1}((1+s)/2) \to 2^{-1} \qquad (s \to 0).$$

The term with $e(-m^2z)$ appears in both $E(0, \mathrm{id}, 4D)$ and $E(0, \chi_D, 4D)$. By Lemma 2.1 and (2.17), (2.18) and (2.19), we have

$$C(-m^2, 0, \mathrm{id}, 4D) = (4D)^{-1}(1-i) \qquad (m \geqslant 0).$$

Therefore we obtain

$$\begin{aligned} &E(0,\mathrm{id},4D) - (4D)^{-1}(1-i)E'(0,\chi_D,4D) \\ &\quad = 1 - 4\pi(1+i)\sum_{n=1}^{\infty} L_{4D}(2,\mathrm{id})^{-1}L_{4D}(1,\chi_{-n})\beta(n,0,\chi_D,4D) \\ &\quad\quad \times \left\{ \prod_{p|2D} A(p,n) - (1-i)(4D)^{-1} \right\} n^{1/2}e(nz). \end{aligned} \tag{2.21}$$

This is the form $f_1(\mathrm{id}, 4D)$. Similarly for a positive divisor l of D we have

$$\begin{aligned} &E(0,\chi_l,4D) - (1-i)l^{1/2}(4D)^{-1}E'(0,\chi_{D/l},4D) \\ &\quad = 1 - 4\pi(1+i)l^{1/2}\sum_{n=1}^{\infty} L_{4D}(2,\mathrm{id})^{-1}L_{4D}(1,\chi_{-ln}) \\ &\quad\quad \times \beta(ln,0,\chi_D,4D)\left\{ \prod_{p|2D} A(p,ln) - (1-i)(4D)^{-1} \right\} n^{1/2}e(nz). \end{aligned} \tag{2.22}$$

We denote this form by $f_1(\chi_l, 4D)$. For a positive divisor l of $2D$ we have

(2.23)

$$\begin{aligned} &E(0,\chi_l,8D) - (1-i)l^{1/2}(8D)^{-1}E'(0,\chi_{2D/l},8D) \\ &\quad = 1 - 4\pi(1+i)l^{1/2}\sum_{n=1}^{\infty} L_{8D}(2,\mathrm{id})^{-1}L_{8D}(1,\chi_{-ln})\beta(ln,0,\chi_{2D},8D) \\ &\quad\quad \times \left\{ A'(2,ln)\prod_{p|D} A(p,ln) - (1-i)(8D)^{-1} \right\} n^{1/2}e(nz), \end{aligned}$$

where

$$A'(2, n) = \sum_{\nu=3}^{\infty} 2^{-3\nu/2} a(2^{\nu}, n).$$

We denote this form by $f_1(\chi_l, 8D)$.

So far we took $s = 0$ in $E(s, \omega, N)$ and $E'(s, \omega, N)$. We can take also $s = -1$ in $E(s, \omega, N)$ and $E'(s, \omega, N)$ to find holomorphic forms. Namely we put

$$f_2^*(\omega, N) = -\frac{E'(s, \omega\chi_N, N) L_N(2s + 2, \omega')}{2\pi(1 + i)N^{1/2} L_N(2s + 1, \omega')}\bigg|_{s=-1},$$

$$f_2(\omega, N) = -\frac{E(s, \bar{\omega}, N) L_N(2s + 2, \bar{\omega}')}{2\pi(1 + i) L_N(2s + 1, \bar{\omega}')}\bigg|_{s=-1}.$$

Noticing that $L(0, \omega) = 0$ if $\omega(-1) = 1$ and $t_n(y, 1, -1/2) = -2\pi(1 + i) y^{1/2} e^{-2\pi n y}$ $(n \geqslant 0)$ we obtain

$$f_2^*(\omega, N) = 1 + \sum_{n=1}^{\infty} \frac{L_N\big(0, (\omega\chi_N)^{(n)}\big)}{L_N(-1, \omega')} \beta(n, -1, \omega\chi_N, N) e(nz), \tag{2.24}$$

(2.25)

$$\begin{aligned} f_2(\omega, N) = {} & C(0, -1, \bar{\omega}, N) \\ & + \sum_{n=1}^{\infty} \frac{L_N\big(0, (\bar{\omega}\chi_N)^{(n)}\big)}{L_N(-1, \bar{\omega}')} \beta(n, -1, \bar{\omega}\chi_N, N) C(n, -1, \bar{\omega}, N) e(nz). \end{aligned}$$

We use these forms later only in some special cases.

Let us put

$$\lambda(n, 4D) = L_{4D}(2, \mathrm{id})^{-1} L_{4D}(1, \chi_{-n}) \beta(n, 0, \chi_D, 4D). \tag{2.26}$$

LEMMA 2.2. *If m is a positive divisor of D, we have*

$$\begin{aligned} f_2^*(\mathrm{id}, 4m) = {} & 1 - 4\pi(1 + i) \sum_{n=1}^{\infty} \lambda(n, 4D)\big(A(2, n) - 4^{-1}(1 - i)\big) \\ & \times \prod_{p \mid m} \big(A(p, n) - p^{-1}\big) \prod_{p \mid D/m} (1 + A(p, n)) n^{1/2} e(nz), \end{aligned} \tag{2.27}$$

(2.28)

$$\begin{aligned} & -2^{-1}(1 + i)\mu(m) f_2(\mathrm{id}, 8m) \\ & \quad = 1 - 4\pi(1 + i) \sum_{\substack{n=1 \\ n \equiv 0 \text{ or } 3 \pmod 4}}^{\infty} \lambda(n, 4D)\big(A(2, n) - 4^{-1}(1 - i)\big) \\ & \quad \times \prod_{p \mid m} \big(A(p, n) - p^{-1}\big) \prod_{p \mid D/m} (1 + A(p, n)) n^{1/2} e(nz). \end{aligned}$$

PROOF. Let $n = ab^2$ where a, b are positive integers and a is square-free. For every prime factor p of n, $h(p)$ denotes the highest exponent such that $p^{h(p)} \mid n$. Using the functional equation of L-function we have

$$\begin{aligned} L_{4m}(-1,\mathrm{id})^{-1}L_{4m}(0,\chi_{-n}) &= -L_{4D}(2,\mathrm{id})^{-1}L_{4D}(1,\chi_{-n}) \\ &\quad\times 2\pi r^{1/2}\prod_{p|2m}\left(1-\left(\frac{-a}{p}\right)\right)\left(1-p^{-1}\left(\frac{-a}{p}\right)\right)^{-1}(1-p)^{-1}(1-p^{-2}) \\ &\quad\times\prod_{p|D/m}\left(1-p^{-1}\left(\frac{-a}{p}\right)\right)^{-1}(1-p^{-2}), \end{aligned}$$

where r is the conductor of χ_{-a}, i.e. $r = 4a$ if $2 \mid a$ or $a \equiv 1 \pmod 4$, $r = a$ if $a \equiv 3 \pmod 4$. By (2.9) we have, putting $h = h(p)$,

$$\begin{aligned} \beta(ab^2,-1,\chi_m,4m) &= \prod_{p|a,\, p\nmid 2m}\left(\sum_{l=0}^{(h-1)/2}p^l\right)\prod_{p|b,\, p\nmid 2ma}\left(\sum_{l=0}^{h/2}p^l-\left(\frac{-a}{p}\right)\sum_{l=0}^{(h/2)-1}p^l\right) \\ &= \prod_{p|a,\, p\nmid 2m}p^{(h-1)/2}\left(\sum_{l=0}^{(h-1)/2}p^{-l}\right)\prod_{p|b,\, p\nmid 2ma}p^{h/2}\left(\sum_{l=0}^{h/2}p^{-l}-\left(\frac{-a}{p}\right)\sum_{l=1}^{h/2}p^{-l}\right) \\ &= \prod_{p|a,\, p\nmid 2D}p^{(h-1)/2}\prod_{p|b,\, p\nmid 2Da}p^{h/2}\prod_{p|a,\, p|D/m}\left(\sum_{l=0}^{(h-1)/2}p^l\right) \\ &\quad\times\prod_{p\nmid a,\, p|b,\, p|D/m}\left(\sum_{l=0}^{h/2}p^l-\left(\frac{-a}{p}\right)\sum_{l=0}^{(h/2)-1}p^l\right)\beta(ab^2,0,\chi_D,4D). \end{aligned}$$

Equality (2.27) can be proved after a straightforward computation.

By Lemma 2.1 and (2.15), (2.16), we can show that $c(0,-1,\mathrm{id},8m) = (i-1)\mu(m)$ and, for $n \neq 0$,

$$c(n,-1,\mathrm{id},8m) = \begin{cases} 0 & \text{if } n \equiv 1,2 \pmod 4, \\ (i-1)\mu(m) & \text{if } \prod_{p|2m}\left(1-\left(\frac{-a}{p}\right)\right) \neq 0. \end{cases}$$

Note that if $\prod_{p|2m}(1-(\frac{-a}{p})) = 0$, then

$$\left(A(2,n)-4^{-1}(1-i)\right)\prod_{p|m}\left(A(p,n)-p^{-1}\right) = 0$$

from (2.17) and (2.18). Moreover we have

$$\frac{L_{8D}(0,\chi_{-n})}{L_{8D}(-1,\mathrm{id})}\beta(n,-1,\chi_{8D},8D) = \frac{L_{4D}(0,\chi_{-n})}{L_{4D}(-1,\mathrm{id})}\beta(n,-1,\chi_D,4D).$$

Now equality (2.28) can be proved easily.

LEMMA 2.3. *$E(\omega, N)$, $f_2^*(\omega, N)$, $E'(\bar{\omega}\chi_N, N)$ and $f_2(\omega, N)$ belong to $\mathfrak{S}(N,\omega)$; $f_1(\mathrm{id},4D)$ belongs to $\mathfrak{S}(4D,\mathrm{id})$; $f_1(\mathrm{id},8D)$ belongs to $\mathfrak{S}(8D,\mathrm{id})$.*

PROOF. We need only to prove this assertion for $E(\omega, N)$. For other cases the proof is similar. By (2.2) we have

$$E(\omega, N)(\gamma(z)) = \omega(d)j(\gamma, z)^3 E(\omega, N)(z)$$

for all $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Using $|L_N(1,(\omega\chi_N)^{(n)})| \leq \rho t^{3/2}$ where $n = tm^2$, t is square-free, ρ is a constant which does not depend on n [**6**, Lemma 5]. We have

$$E(\omega, N)(z) \leq 1 + \rho \sum_{n=1}^{\infty} t^{3/2}\Big(\sum_{ab|m} 1\Big)e^{-2\pi n y} \leq 1 + \rho \sum_{n=1}^{\infty} n^{3/2}e^{-2\pi n y} \leq 1 + \rho y^{-4},$$

where ρ can be different constant. Therefore $E(\omega, N)$ is holomorphic on H. Furthermore for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ and $c \neq 0$ we have

$$\left|E(\omega, N)(\gamma(z))(cz + d)^{-3/2}\right| \leq \left(1 + \rho y^{-4}|cz + d|^8\right)|cz + d|^{-3/2} \leq \rho y^{5/2}$$

when $y \to +\infty$. This shows that $E(\omega, N)$ is holomorphic at every cusp. Hence it is an integral form.

There is a standard argument (originally due to Petersson) which shows that $E(\omega, N)$ is orthogonal to $S(N, \omega)$. For the reader's convenience, we give a proof here.

Let $f(z) = \sum_{n=1}^{\infty} c_n e(nz)$ be a cusp form in $S(N, \omega)$. Since $\int_0^1 \bar{f}(x + iy)\, dx = 0$, and

$$\bar{f}(\gamma(z))\mathrm{Im}(\gamma(z))^{(s+3)/2} = \bar{\omega}(d_\gamma)j(\gamma, z)^{-3}|j(\gamma, z)|^{-2s}\bar{f}(z)y^{(s+3)/2}$$

for all $\gamma \in \Gamma_0(N)$, we have

$$0 = \int_0^{\infty} y^{(s+3)/2-2}\int_0^1 \bar{f}(x + iy)\, dx\, dy$$

$$= \int_{\Gamma_0(N)\backslash H} E(s, \bar{\omega}, N)\bar{f}(x + iy)y^{-1/2}\, dx\, dy.$$

Here we use the fact that $\{0 \leq x < 1,\ 0 < y < +\infty\}$ is a fundamental domain of Γ_∞. Taking $s = 0$ we prove that $E(\omega, N)$ is orthogonal to $S(N, \omega)$.

3. Some operators on $M(N, \omega)$. Throughout this section $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ denotes a form in $M(N, \omega)$.

1. The *shift* $V(m) = m^{-3/4}\{\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right), m^{-1/4}\}$ is defined by $(f|V(m))(z) = f(mz)$, where m is a positive integer. By Proposition 1.3 of [**6**] we know that $f|V(m)$ belongs to $M(mN, \omega\chi_m)$.

2. *Hecke operators*. Let $\Gamma_0 = \Gamma_0(N)$ with $4 | N$. Δ_0 is the image of Γ_0 in G under the map $\gamma \mapsto \gamma^* = \{\gamma, j(\gamma, z)\}$. Put

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad \xi = \{\alpha, d^{1/4}\} \qquad (d \in \mathbf{Z},\ d > 0).$$

Suppose $\Delta_0\xi\Delta_0 = \bigcup_{j=1}^m \Delta_0\xi_j$ (disjoint), $\Gamma_0\alpha\Gamma_0 = \bigcup_{j=1}^m \Gamma_0\alpha_j$ with $\alpha_j = P(\xi_j)$, where P is the map $\{\gamma, \phi(\gamma)\} \mapsto \gamma$ from G onto $GL_2^+(\mathbf{R})$. Then we define a linear operator $T(d)$ on $M(N, \omega)$ by

$$f|T(d) = d^{-1/4}\sum_{j=1}^{m} \omega(a_j)f|\xi_j,$$

where $\alpha_j = \binom{a_j\ *}{*\ \ *}$. It can easily be seen that $T(d)$ maps $M(N, \omega)$ and $S(N, \omega)$ into themselves, when d is a square. Let p be a prime and $(f \mid T(p^2))(z) = \sum_{n=0}^{\infty} b(n)e(nz)$. Then we have [6, Theorem 1.7],

$$b(n) = a(p^2n) + \omega(p)\left(\frac{-n}{p}\right)a(n) + \omega(p^2)pa(n/p^2). \tag{3.1}$$

We understand that $a(n/p^2) = 0$ if n is not divisible by p^2. In particular, if $p \mid N$ then $b(n) = a(p^2n)$.

3. *Normalizer of* $\Gamma_0(N)$. We know that the matrix $\binom{Q\ \ -1}{uN\ \ vQ}$, with $u, v, Q \in \mathbf{Z}$, $Q > 0$, $Q \mid N$, $vQ + uN/Q = 1$, belongs to the normalizer of $\Gamma_0(N)$. Let us define an element $W(Q)$ in G as follows. Put $\gamma^* = \{\gamma, j(\gamma, z)\}$ for $\gamma \in \Gamma_0(4)$. If $2 \nmid Q$, we define

$$\begin{aligned} W(Q) &= \left\{\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}, Q^{1/4}\right\} \cdot \begin{pmatrix} Q & -1 \\ uNQ^{-1} & V \end{pmatrix}^* \\ &= \left\{\begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}, \varepsilon_Q^{-1}Q^{1/4}(uNQ^{-1}z + v)^{1/2}\right\}. \end{aligned}$$

If $4 \mid Q$, we define

$$\begin{aligned} W(Q) &= \left\{\begin{pmatrix} 0 & -1 \\ Q & 0 \end{pmatrix}, Q^{1/4}(-iz)^{1/2}\right\} \cdot \begin{pmatrix} uQ^{-1}N & v \\ -Q & 1 \end{pmatrix}^* \\ &= \left\{\begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}, e^{-\pi i/4}Q^{1/4}(uNQ^{-1}z + v)^{1/2}\right\}. \end{aligned}$$

The element $W(Q)$ depends on the choice of u and v. We have, however,

LEMMA 3.1. *Let $\omega = \omega_1\omega_2$ where ω_1, ω_2 are characters modulo Q and N/Q respectively. Then the form $g = f \mid W(Q)$ does not depend on u and v, and belongs to $M(N, \bar{\omega}_1\omega_2\chi_Q)$.*

PROOF. We consider the case $2 \nmid Q$. The proof for the case $4 \mid Q$ is similar. Suppose u_1, v_1 also satisfy $v_1Q + u_1N/Q = 1$. Our first assertion follows from the relation

$$\begin{aligned} &\left\{\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}, Q^{1/4}\right\} \cdot \begin{pmatrix} Q & -1 \\ uNQ^{-1} & v \end{pmatrix}^* \begin{pmatrix} v_1 & 1 \\ -u_1NQ^{-1} & Q \end{pmatrix}^* \left\{\begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix}, Q^{-1/4}\right\} \\ &\qquad = \begin{pmatrix} 1 & 0 \\ (uv_1 - u_1v)N & 1 \end{pmatrix}^*. \end{aligned}$$

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \qquad \alpha = \begin{pmatrix} Q & -1 \\ uNQ^{-1} & v \end{pmatrix}$$

and

$$\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix} \gamma \begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}^{-1}.$$

Then we see easily that $N \mid c_0$, $d_0 \equiv a \pmod{4Q}$ and $d_0 \equiv d \pmod{N/Q}$. Therefore we obtain

$$W(Q)\gamma^* W(Q)^{-1} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}, Q^{1/4}\right\}(\alpha\gamma\alpha^{-1})^* \left\{\begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix}, Q^{-1/4}\right\} = \gamma_0^*\left\{1, \left(\frac{Q}{d}\right)\right\};$$

hence

$$g|\gamma^* = f|W(Q)\gamma^* W(Q)^{-1} W(Q) = \omega\chi_Q(d_0)g = \bar{\omega}_1\omega_2\chi_Q(d)g.$$

This completes the proof.

4. *Other operator.* For a given form $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in M(N, \omega)$ and a Dirichlet character ψ with conductor m, we define a form $h(z)$ by

$$\begin{aligned} h(z) &= \sum_{u=1}^{m} \bar{\psi}(u) f(z + u/m) \\ &= \sum_{u=1}^{m} \bar{\psi}(u) e(u/m) \sum_{n=0}^{\infty} \psi(n) a(n) e(nz). \end{aligned}$$

LEMMA 3.2. *Let s be the conductor of ω. Then $h(z)$ defined above belongs to $M(N^*, \omega\psi^2)$ where N^* is the least common multiple of N, sm, $4m$ and m^2.*

PROOF. This proof is similar to those in the case of integral weight (cf. [**8**, Proposition 3.64]). It is sufficient to check the behavior of $h(z)$ under the operator $h \to h \mid \gamma^*$ where $\gamma = \left(\begin{smallmatrix} a & b \\ cN^* & d \end{smallmatrix}\right) \in \Gamma_0(N^*)$. Put

$$\begin{aligned} a' &= a + cuN^*/m, \\ b' &= b + du(1 - ad)/m - cd^2u^2N^*/m^2, \\ d' &= d - cd^2uN^*/m. \end{aligned}$$

Then a', b', d' are integers, $d' \equiv d \pmod{s}$ and $d' \equiv d \pmod{4}$. We can verify that

$$\left\{\begin{pmatrix} 1 & u/m \\ 0 & 1 \end{pmatrix}, 1\right\} \cdot \gamma^* = \begin{pmatrix} a' & b' \\ cN^* & d' \end{pmatrix}^* \cdot \left\{\begin{pmatrix} 1 & d^2u/m \\ 0 & 1 \end{pmatrix}, 1\right\}.$$

Therefore we have

$$\begin{aligned} h \mid \gamma &= \sum_{u=1}^{m} \bar{\psi}(u) f \left| \left\{\begin{pmatrix} 1 & u/m \\ 0 & 1 \end{pmatrix}, 1\right\} \cdot \gamma^* \right. \\ &= \omega(d) \sum_{u=1}^{m} \bar{\psi}(u) f \left| \left\{\begin{pmatrix} 1 & d^2u/m \\ 0 & 1 \end{pmatrix}, 1\right\} \right. = \omega\psi^2(d) h. \end{aligned}$$

This proves the lemma.

We have seen above that $E(\omega, N)$ $(\omega^2 \neq \mathrm{id})$, $f_1(\mathrm{id}, 4D)$ and $f_1(\mathrm{id}, 8D)$, which we mentioned in the theorem, are values of $E(s, \omega, N)$ and $E'(s, \omega, N)$ at $s = 0$. However in the following proof of the theorem in §5 and §6, we shall use $f_2^*(\mathrm{id}, 4D)$ and $f_2(\mathrm{id}, 8D)$ which are the values of $E'(s, \chi_D, 4D)$ and $E(s, \mathrm{id}, 8D)$ at $s = -1$. Now we point out that $f_2^*(\mathrm{id}, 4D)$ and $f_2(\mathrm{id}, 8D)$ can be obtained from $f_1(\mathrm{id}, 4D)$ and $f_1(\mathrm{id}, 8D)$ by applying Hecke operators. Therefore to use them in the proof of the theorem is justifiable.

For $m \mid D$, put

$$h_m(z) = 1 - 4\pi(1+i)\sum_{n=1}^{\infty} \lambda(n,4D)\big(A(2,n) - 4^{-1}(1-i)\big) \times \prod_{p\mid m}\big(A(p,n) - p^{-1}\big)\prod_{p\mid D/m} A(p,n)n^{1/2}e(nz).$$

Then we have, by (2.20),

$$f_1(\mathrm{id},4D) \mid T(4) = 2f_1(\mathrm{id},4D) - h_1(z),$$
$$h_m(z) \mid T(p^2) = ph_m(z) + (1-p)h_{mp}(z) \qquad (mp \mid D).$$

This proves our assertion for $f_2^*(\mathrm{id},4D) = h_D(z)$.

For $m \mid D$, put

$$h'_m(z) = 1 - 4\pi(1+i)\sum_{n=1}^{\infty} \lambda(n,4D)A'(2,n) \times \prod_{p\mid m}\big(A(p,n) - p^{-1}\big)\prod_{p\mid D/m} A(p,n)n^{1/2}e(nz).$$

Then similarly we have, by (2.20),

$$f_1(\mathrm{id},8D) \mid T(p^2) = pf_1(\mathrm{id},8D) + (1-p)h'_p(z) \qquad (p \mid D),$$
$$h'_m(z) \mid T(p^2) = ph'_m(z) + (1-p)h'_{mp}(z) \qquad (mp \mid D).$$

It is easy to verify by (2.15) that, if $n \neq 0$,

$$A'(2,n) - (1-i)/8 = \begin{cases} A(2,n) - (1-i)/4 & \text{if } n \equiv 0, 3 \pmod 4, \\ 3\big(A(2,n) - (1-i)/4\big) & \text{if } n \equiv 1, 2 \pmod 4. \end{cases}$$

Hence

$$\begin{aligned} h'_D(z) &= 1 - 4\pi(1+i)\sum_{n=1}^{\infty}\lambda(n,4D)\big(A'(2,n) - (1-i)/8\big) \\ &\quad \times \prod_{p\mid D}\big(A(p,n) - p^{-1}\big)n^{1/2}e(nz) \\ &\quad - \pi\sum_{n=1}^{\infty}\lambda(n,4D)\prod_{p\mid D}\big(A(p,n) - p^{-1}\big)n^{1/2}e(nz) \\ &= 3f_2^*(\mathrm{id},4D) + (1+i)\mu(D)f_2(\mathrm{id},8D) - 2^{-1}g(\mathrm{id},D,4D). \end{aligned}$$

Therefore our assertion is proved for $f_2(\mathrm{id},8D)$. (g is defined in §5, see Lemma 5.2.)

4. The values of Eisenstein series at cusps. From now on, when we say d/c is a cusp, it always means that $d, c \in \mathbf{Z}$, $c > 0$ and $(c,d) = 1$. Let $f(z)$ belong to $M(N,\omega)$ and d/c be a cusp. Then there exists an element $\rho = \left(\begin{smallmatrix} a & b \\ -c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ such that $\rho(d/c) = i\infty$. We call the constant term of the Fourier expansion of $f \mid \{\rho^{-1}, (cz+a)^{1/2}\}$ at $z = i\infty$ *the value of f at the cusp d/c*, and denote it by

$V(f, d/c)$. This value is independent of the choice of a and b. Furthermore we have

$$\begin{aligned} V(f, d/c) &= \lim_{z \to i\infty} f((dz - b)/(cz + a))(cz + a)^{-3/2} \\ &= \lim_{z \to i\infty} f\left(-c^{-1}(cz + a)^{-1} + d/c\right)(cz + a)^{-3/2} \\ &= \lim_{\tau \to 0} (-c\tau)^{3/2} f(\tau + d/c). \end{aligned} \tag{4.1}$$

Obviously we have $V(f, i\infty) = \lim_{z \to i\infty} f(z)$.

For every positive divisor $c = \prod_{p|N} p^{c(p)}$ of N we put $g(c) = \phi((c, N/C))$ where ϕ is Euler's function. Suppose $\{d_1, d_2, \ldots, d_{g(c)}\}$ is a full set of representatives of $(\mathbf{Z}/(c, N/c))^*$. Then the set

$$S(N) = \{d_i/c | 1 \leqslant i \leqslant g(c),\ c|N\}$$

is a full set of representatives of $\Gamma_0(N)$-equivalence classes of cusps. The number of cusps in $S(N)$ is $\sum_{c|N} \phi((c, N/c))$. When we consider the values of a form in $M(N, \omega)$ at cusps, it is sufficient to consider its values at the cusps in $S(N)$ (see Lemma 4.2 below).

We fix an order of the prime factors of N as $p_0 = 2, p_1, p_2, \ldots, p_\nu$. To every divisor c of N we attach an ordered $(\nu + 1)$-tuple $(c(p_0), c(p_1), \ldots, c(p_\nu))$. Arrange these $(\nu + 1)$-tuples in the lexicographical order. Thus we can define an order among the divisors c of N according to the order of $(c(p_0), c(p_1), \ldots, c(p_\nu))$. We write $c_1 \prec c_2$ to mean that c_1 precedes c_2.

We recall that if cusps d_1/c_1 and d_2/c_2 are $\Gamma_0(N)$-equivalent, then $(c_1, N) = (c_2, N)$.

LEMMA 4.1. *If $f \in M(N, \omega)$ and d/c is a cusp such that $N | c$, then $V(f, d/c) = \bar{\omega}(d)\chi_c(d)\varepsilon_d V(f, i\infty)$.*

This follows immediately from the definition.

LEMMA 4.2. *If $f \in M(N, \omega)$ and cusp $s_1 = d_1/c_1$ is equivalent to cusp $s_2 = d_2/c_2$, i.e. if there exists an element $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ such that $\rho(s_1) = s_2$, then*

$$V(f, s_2) = \omega\chi_c(d)\varepsilon_d V(f, s_1).$$

PROOF. Let

$$\rho_2 = \begin{pmatrix} a_2 & b_2 \\ -c_2 & d_2 \end{pmatrix} \in SL_2(\mathbf{Z}) \quad \text{and} \quad \rho_1 = \rho_2 \rho.$$

From $\rho^{-1}(s_2) = s_1$ we have $c_1 = -cd_2 + ac_2,\ d_1 = dd_2 - bc_2$, hence

$$\rho_1 = \begin{pmatrix} a_1 & b_1 \\ -c_1 & d_1 \end{pmatrix},$$

where a_1, b_1 are integers. Since c_1 and c_2 are all positive, we have

$$\{\rho_1^{-1}, (c_1 z + a_1)^{1/2}\} = \{\rho^{-1}, (-cz + a)^{1/2}\} \cdot \{\rho_2^{-1}, (c_2 z + a_2)^{1/2}\},$$

and therefore

$$f|\{\rho_1^{-1},(c_1z+a_1)^{1/2}\}=\omega(a)\chi_{-c}(a)\varepsilon_a f|\{\rho_2^{-1},(c_2z+a_2)^{1/2}\}$$
$$=\bar{\omega}\chi_c(d)\varepsilon_a^{-1}f|\{\rho_2^{-1},(c_2z+a_2)^{1/2}\}.$$

This implies the lemma.

In the next several lemmas we discuss the values at cusps of the Eisenstein series defined in §2.

LEMMA 4.3. *For* $E'(\omega, N)$ $(\omega^2 \neq \mathrm{id})$ *we have* $V(E'(\omega, N), 1) = i$ *and* $V(E'(\omega, N), d/c) = 0$ *for any* $d/c \in S(N)$ *with* $c \neq 1$.

PROOF. By the definition of $E(\omega, N)$ and $E'(\omega, N)$ we have

$$(-z)^{3/2}E'(\omega, N)(z) = i \cdot E(\omega, N)(-1/Nz).$$

Hence $V(E', 1) = iV(E, i\infty) = i$ by (4.1) and (2.13). Here we put $E = E(\omega, N)$, $E' = E'(\omega, N)$. Let α be a positive integer such that $\alpha \neq 1$, $\alpha \mid N$ and $(\alpha, N/\alpha) = 1$. Suppose $\omega = \omega_1\omega_2$ where ω_1, ω_2 are characters modulo α and N/α respectively. Let p be a prime factor of α. From (2.14) we see easily that $E' \mid T(p^2) = pE'$, therefore we obtain

$$pE'(z + (1/\alpha)) = p^{-2}\sum_{k=1}^{p^2} E'(p^{-2}z + (1 + k\alpha)/\alpha p^2).$$

For any k, $1 + k\alpha$ is prime to αp^2, hence

$$pV(E', 1/\alpha) = p^{-2}\sum_{k=1}^{p^2} V(E', (1 + k\alpha)/\alpha p^2). \tag{4.2}$$

The cusp $(1 + k\alpha)/\alpha p^2$ is $\Gamma_0(N)$-equivalent to $1/\alpha$, i.e. there exists an element $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(N)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 + k\alpha \\ \alpha p^2 \end{pmatrix}.$$

We can find that $a \equiv d \equiv 1 \pmod{\alpha}$ and $d \equiv p^2 \pmod{N/\alpha}$. In both cases $4 \mid \alpha$ and $2 \nmid \alpha$, we always have $\varepsilon_d = 1$ and $(\frac{c}{d}) = 1$. Hence $V(E', (1 + k\alpha)/\alpha p^2) = \omega_2(p^2)V(E', 1/\alpha)$. This implies $V(E', 1/\alpha) = 0$ from (4.2). Now suppose β is a positive divisor of N, $\beta \neq 1$. Put $m = \prod_{p|\beta} p$. We can find a positive integer l such that $((m^{2l}\beta, N), N/(m^{2l}\beta, N)) = 1$. Using $E' \mid T(m^{2l}) = m^l E'$, we obtain

$$m^l V(E', d/\beta) = m^{-2l}\sum_{k=1}^{2l} V(E', (d + k\beta)/m^{2l}\beta) = 0$$

because cusp $(d + k\beta)/m^{2l}\beta$ is $\Gamma_0(N)$-equivalent to a cusp $1/\alpha$ of the above type. This completes the proof.

The next lemma follows from Lemma 4.3 immediately.

LEMMA 4.4. $V(E(\omega, N), i\infty) = 1$ *and* $V(E(\omega, N), d/c) = 0$ *for any* $d/c \in S(N)$ *with* $c \neq N$.

LEMMA 4.5.

$$V(f_1(\mathrm{id},4D),1) = -(1+i)(4D)^{-1}, \qquad V(f_1(\mathrm{id},8D),1) = -(1+i)(8D)^{-1}.$$

PROOF. By definition we have

$$f_1(\mathrm{id},4D)(z) = E(0,\mathrm{id},4D)(z) - (1-i)(4D)^{-1}E(0,\chi_D,4D)(-1/4Dz)z^{-3/2}.$$

Hence

$$\begin{aligned} f_1(\mathrm{id},4D)(-1/4Dz)z^{-3/2} &= E'(0,\mathrm{id},4D)(z) - 2D^{1/2}(1+i)E(0,\chi_D,4D)(z) \\ &= -2D^{1/2}(1+i)f_1(\chi_D,4D)(z). \end{aligned}$$

Therefore we obtain

$$V(f_1(\mathrm{id},4D),1) = \lim_{z\to i\infty}(4Dz)^{-3/2}f_1(\mathrm{id},4D)(-1/4Dz) = -(1+i)(4D)^{-1}.$$

The other equality can be proved similarly.

In the rest of this section, m, l and β always denote positive divisors of D, and α is a positive divisor of m.

LEMMA 4.6. *Suppose $f(z)$ belongs to $\mathfrak{S}(8D, \chi_l)$ and satisfies*

$$f\,|\,T(p^2) = f \quad (p\,|\,m), \qquad f\,|\,T(p^2) = pf \quad (p\,|\,D/m).$$

Then we have

$$V(f,1/\alpha) = \mu(\alpha)\alpha(\alpha,l)^{-1/2}\varepsilon^{-1}_{\alpha/(\alpha,l)}\left(\frac{l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f,1),$$

$$V(f,1/4\alpha) = \mu(\alpha)\alpha(\alpha,l)^{-1/2}\varepsilon_{l/(\alpha,l)}\varepsilon_l^{-1}\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/4),$$

$$V(f,1/8\alpha) = \mu(\alpha)\alpha(\alpha,l)^{-1/2}\varepsilon_{l/(\alpha,l)}\varepsilon_l^{-1}\left(\frac{2}{(l,\alpha)}\right)\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/8).$$

Moreover, $f(z)$ vanishes at $1/2^r\beta$ if $(\beta, D/m) \neq 1$, $r = 0,2,3$.

PROOF. The last assertion can be proved in the same fashion as we did in Lemma 4.3. The equality for $V(f,1/\alpha)$ is obvious for $\alpha = 1$. Assuming it is true for $V(f,1/\alpha)$, let us prove it for $V(f,1/\alpha p)$ with $\alpha p\,|\,m$. By means of $f\,|\,T(p^2) = f$ we have

$$f(z+1/\alpha) = p^{-2}\sum_{k=1}^{p^2} f(p^{-2}z + p^{-2}\alpha^{-1}(1+k\alpha)).$$

There exists a unique k_1 such that $1 \leqslant k_1 \leqslant p$, $1+\alpha k_1 = pt_1$; a unique k_2 such that $1 \leqslant k_2 \leqslant p^2$, $1 + k_2\alpha = p^2t_2$, where t_1 and t_2 are integers. Hence we obtain

$$\begin{aligned} V(f,1/\alpha) = {} & p^{-2}\sum_{1\leqslant k\leqslant p^2,\, p\nmid 1+k\alpha} V(f,(1+k\alpha)/\alpha p^2) \\ & + p^{-1/2}\sum_{1\leqslant k\leqslant p,\, p\nmid t_1+k\alpha} V(f,(t_1+k\alpha)/\alpha p) + pV(f,t_2/\alpha). \end{aligned} \tag{4.3}$$

The cusps $(1+k\alpha)/\alpha p^2$, $(t_1+k\alpha)/\alpha p$ and t_2/α are $\Gamma_0(8D)$-equivalent to $1/\alpha p$, $1/\alpha p$ and $1/\alpha$ respectively. We suppose $p \nmid l$ first, then using Lemma 4.2, we find

$$V(f,(1+\alpha k)/\alpha p^2) = \varepsilon_{\alpha p/(\alpha,l)}\varepsilon^{-1}_{\alpha/(\alpha,l)}\left(\frac{l/(\alpha,l)}{p}\right)V(f,1/\alpha p),$$
$$V(f,(t_1+\alpha k)/\alpha p) = \left(\frac{t_1+\alpha k}{p}\right)\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha p),$$
$$V(f,t_2/\alpha) = V(f,1/\alpha).$$

Combining these with (4.3), we obtain

$$V(f,1/\alpha p) = -p\varepsilon_{\alpha/(\alpha,l)}\varepsilon^{-1}_{\alpha p/(\alpha p,l)}\left(\frac{l/(\alpha,l)}{p}\right)V(f,1/\alpha).$$

This implies the equality we desired.

If $p \mid l$, we find

$$V(f,(1+\alpha k)/\alpha p^2) = \varepsilon_{\alpha/(\alpha,l)}\varepsilon^{-1}_{\alpha p/(\alpha,l)}\left(\frac{(1+\alpha k)l/(\alpha p,l)}{p}\right)V(f,1/\alpha p),$$
$$V(f,(t_1+\alpha k)/\alpha p) = \left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha p),$$
$$V(f,t_2/\alpha) = V(f,1/\alpha).$$

Then from (4.3) we obtain

$$V(f,1/\alpha p) = -p^{-1/2}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha).$$

This completes the proof of the equality for $V(f,1/\alpha)$. The other two equalities can be proved in the same way.

Similarly we can prove

LEMMA 4.7. *Suppose $f(z)$ belongs to $\mathfrak{S}(8D,\chi_{2l})$ and satisfies the same conditions as in Lemma 4.6. Then we have*

$$V(f,1/2^r\alpha) = \mu(\alpha)\alpha(\alpha,l)^{-1/2}\varepsilon^{-1}_{\alpha/(\alpha,l)}\left(\frac{2^{1-r}l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f,1/2^r) \qquad (r=0,1),$$
$$V(f,1/8\alpha) = \mu(\alpha)\alpha(\alpha,l)^{-1/2}\varepsilon_{l/(\alpha,l)}\varepsilon_l^{-1}\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/8),$$

and $f(z)$ vanishes at $1/2^r\beta$ if $(\beta, D/m)\neq 1$, $r=0,1,3$.

LEMMA 4.8.

$$V(f_2^*(\mathrm{id},4D),1/\beta) = -4^{-1}(1+i)\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1},$$
$$V(f_2^*(\mathrm{id},4D),1/2\beta) = 0,$$
$$V(f_2^*(\mathrm{id},4D),1/4\beta) = \mu(D/\beta)\beta D^{-1}.$$

PROOF. We see easily from (2.24) that $f_2^*(\mathrm{id}, 4D) \mid T(p^2) = f_2^*(\mathrm{id}, 4D)$ for any prime factor p of $2D$. Now we show $V(f_2^*(\mathrm{id}, 4D), 1/2\beta) = 0$ for any β. Taking $p = 2$ we have

$$f_2^*(\mathrm{id}, 4D)(z + (1/2\beta)) = 4^{-1} \sum_{k=1}^{4} f_2^*(\mathrm{id}, 4D)((z/4) + (1 + 2\beta k)/8\beta).$$

The cusp $(1 + 2\beta k)/8\beta$ is $\Gamma_0(4D)$-equivalent to $1/4\beta$ for any k. By virtue of Lemma 4.2, we have

$$\begin{aligned} V(f_2^*(\mathrm{id}, 4D), 1/2\beta) &= 4^{-1} \sum_{k=1}^{4} V(f_2^*(\mathrm{id}, 4D), (1 + 2\beta k)/8\beta) \\ &= 4^{-1} \sum_{k=1}^{4} \left(\frac{2\beta}{1 + 2\beta k}\right) \varepsilon_{1+2k} V(f_2^*(\mathrm{id}, 4D), 1/4\beta) = 0. \end{aligned}$$

Since $V(f_2^*(\mathrm{id}, 4D), 1/4D) = 1$, we have $V(f_2^*(\mathrm{id}, 4D), 1/4) = \mu(D)D^{-1}$ by Lemma 4.6. Using $f_2^*(\mathrm{id}, 4D) \mid T(4) = f_2^*(\mathrm{id}, 4D)$ again, we find

$$V(f_2^*(\mathrm{id}, 4D), 1) = 4^{-1}(1 + i)V(f_2^*(\mathrm{id}, 4D), 1/4) + 2V(f_2^*(\mathrm{id}, 4D), 1).$$

Here we use the fact that cusp $3/4$ is $\Gamma_0(4D)$-equivalent to $1/4$. Therefore

$$V(f_2^*(\mathrm{id}, 4D), 1) = -4^{-1}(1 + i)\mu(D)D^{-1}.$$

Now the lemma follows from Lemma 4.6.

LEMMA 4.9.

$$\begin{aligned} V(f_2^*(\chi_{2D}, 8D), 1/\beta) &= -2^{-3/2}(1 + i)\mu(D/\beta)\beta^{1/2}D^{-1/2}, \\ V(f_2^*(\chi_{2D}, 8D), 1/2\beta) &= 2^{-1}(1 + i)\mu(D/\beta)\beta^{1/2}D^{-1/2}, \\ V(f_2^*(\chi_{2D}, 8D), 1/4\beta) &= 0, \\ V(f_2^*(\chi_{2D}, 8D), 1/8\beta) &= \mu(D/\beta)\beta^{1/2}D^{-1/2}\varepsilon_{D/\beta}. \end{aligned}$$

PROOF. Put $h = f_2^*(\chi_{2D}, 8D)$. Then we see $h \mid T(P^2) = h$ for any $p \mid 2D$ from (2.24). Using the equality for $T(4)$ and the fact that $V(h, 1/8D) = 1$, we can show, as we did in Lemma 4.18, that $V(h, 1/4\beta) = 0$ for any β, and

$$\begin{aligned} V(h, 1) &= -2^{-3/2}(1 + i)\mu(D)D^{-1/2}, \\ V(h, 1/2) &= 2^{-1}(1 + i)\mu(D)D^{-1/2}, \\ V(h, 1/8) &= \mu(D)D^{-1/2}\varepsilon_D. \end{aligned}$$

Then the lemma follows from Lemma 4.7.

LEMMA 4.10.

$$\begin{aligned} V\big(-2^{-1}(1 + i)\mu(D)f_2(\mathrm{id}, 8D), 1/\beta\big) &= -16^{-1}(1 + i)\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1}, \\ V\big(-2^{-1}(1 + i)\mu(D)f_2(\mathrm{id}, 8D), 1/2\beta\big) &= 0, \\ V\big(-2^{-1}(1 + i)\mu(D)f_2(\mathrm{id}, 8D), 1/4\beta\big) &= -2^{-1}\mu(D/\beta)\beta D^{-1}, \\ V\big(-2^{-1}(1 + i)\mu(D)f_2(\mathrm{id}, 8D), 1/8\beta\big) &= \mu(D/\beta)\beta D^{-1}. \end{aligned}$$

PROOF. By definition of $f_2^*(\chi_{2D}, 8D)$ and $f_2(\mathrm{id}, 8D)$, we have

$$f_2^*(\chi_{2D}, 8D)(-1/8Dz)z^{-3/2} = 8Dif_2(\mathrm{id}, 8D)(z).$$

Let c be a positive divisor of $8D$, since

$$(-cz)^{3/2} f_2(\mathrm{id}, 8D)(z + c^{-1}) = -i(8D)^{-1}c^{3/2}f_2^*(\chi_{2D}, 8D) \times \left(cz/\left(8D(z + c^{-1})\right) - (c/8D)\right)\left(-z/\left(z + c^{-1}\right)\right)^{3/2},$$

we have

$$V(f_2(\mathrm{id}, 8D), 1/c) = -i(8D)^{-1}c^{3/2}V(f_2^*(\chi_{2D}, 8D), -c/8D).$$

Now the lemma can be proved by using Lemma 4.9.

Finally we state the next lemma about the operator $W(Q)$ defined in §3 to conclude this section.

LEMMA 4.11. *Suppose an element f of $\mathfrak{S}(N, \omega)$ vanishes at all cusps of $S(N)$ except $1/N$. Then $g = f \mid W(Q)$ vanishes at all cusps of $S(N)$ except $1/NQ^{-1}$.*

PROOF. This follows easily from the relation

$$(Qz - 1)(uNz + Vq)^{-1}\big|_{z=Q/N} = (Q - N/Q)((u - v)N)^{-1}$$

and the fact that the cusp $(Q - N/Q)((u - v)N)^{-1}$ is $\Gamma_0(N)$-equivalent to $1/N$.

5. Proof of Theorem for $\mathfrak{S}(4D, \chi_l)$. Given a positive integer N, put $N = 2^{e(2)}N'$ with odd N'. Let ω be a Dirichlet character modulo N such that $\omega(-1) = 1$, and $r(\omega)$ the conductor of ω. Then we have (cf. Cohen and Oesterlé [1]), if $e(2) = 2$,

$$\dim \mathfrak{S}(N, \omega) = 2 \sum_{\substack{c \mid N' \\ (c, N'/c) \mid N/r(\omega)}} \phi((c, N'/c)) - \dim \mathfrak{S}(N, 1/2, \omega); \tag{5.1}$$

if $e(2) = 3$,

$$\dim \mathfrak{S}(N, \omega) = 3 \sum_{\substack{c \mid N' \\ (c, N'/c) \mid N/r(\omega)}} \phi((c, N'/c)) - \dim \mathfrak{S}(N, 1/2, \omega); \tag{5.2}$$

if $e(2) \geqslant 4$,

$$\dim \mathfrak{S}(N, \omega) = \sum_{\substack{c \mid N \\ (c, N/c) \mid N/r(\omega)}} \phi((c, N/c)) - \dim \mathfrak{S}(N, 1/2, \omega). \tag{5.3}$$

Here ϕ is the Euler function.

By Theorem B in [5] we know that $\dim \mathfrak{S}(N, 1/2, \omega)$ is equal to the number of pairs (ψ, t), where t is an integer $\geqslant 1$, and ψ is a totally even primitive character with conductor $r(\psi)$, such that

$$\text{(5.4)} \quad \begin{array}{ll} \text{(i)} & 4r(\psi)^2 t \quad \text{divides } N, \\ \text{(ii)} & \omega(n) = \psi(n)\chi_t(n) \quad \text{for all } n \text{ prime to } N. \end{array}$$

Using (5.1)–(5.4) we can evaluate $\dim \mathfrak{S}(N, \omega)$.

Throughout this and the next sections, let D denote an odd square-free integer; m, l and β positive divisors of D; α a positive divisor of m; ν the number of prime factors of D. By (5.1) and (5.4) we have

$$\dim \mathfrak{S}(4D, \chi_l) = 2^{\nu+1} - 1. \tag{5.5}$$

Now let us define

$$g(\chi_l, 4D, 4D) = 1 - 4\pi(1+i)l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D)\big(A(2, ln) - 4^{-1}(1-i)\big) \times \prod_{p|D} \big(A(p, ln) - p^{-1}\big)n^{1/2}e(nz),$$

$$g(\chi_l, 4m, 4D) = -4\pi(1+i)l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D)\big(A(2, ln) - 4^{-1}(1-i)\big) \times \prod_{p|m} \big(A(p, ln) - p^{-1}\big)n^{1/2}e(nz)$$

for $m \neq D$. Moreover, for $m \neq 1$, define

$$g(\chi_l, m, 4D) = 2\pi l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D) \prod_{p|m} \big(A(p, ln) - p^{-1}\big)n^{1/2}e(nz),$$

where $\lambda(n, N)$, $A(2, n)$ and $A(p, n)$ are defined in §2.

LEMMA 5.1. *The form $g(\chi_l, 4m, 4D)$ belongs to $\mathfrak{S}(4D, \chi_l)$ and*

$$V(g(\chi_l, 4m, 4D), 1/\alpha) = -4^{-1}(1+i)\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l, \alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right),$$

$$V(g(\chi_l, 4m, 4D), 1/4\alpha) = \mu(m/\alpha)\alpha m^{-1}l^{1/2}(l, \alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right).$$

It vanishes at the other cusps in $S(4D)$.

PROOF. First of all we consider the case $l = 1$. The lemma holds for $g(\mathrm{id}, 4D, 4D) = f_2^*(\mathrm{id}, 4D)$ by Lemma 4.8. If $m \neq D$, we have

$$\begin{aligned} g(\mathrm{id}, 4m, 4D) &= -4\pi(1+i) \prod_{p|D/m} p(1+p)^{-1} \\ &\quad \times \sum_{n=1}^{\infty} \lambda(n, 4D)\big(A(2, n) - 4^{-1}(1-i)\big) \prod_{p|m} \big(A(p, n) - p^{-1}\big) \\ &\quad \times \prod_{p|D/m} \big\{1 + A(p, n) - \big(A(p, n) - p^{-1}\big)\big\}n^{1/2}e(nz) \\ &= \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) f_2^*(\mathrm{id}, 4md) \end{aligned} \tag{5.6}$$

by (2.27). This shows that $g(\mathrm{id}, 4m, 4D)$ belongs to $\mathfrak{S}(4D, \mathrm{id})$. By means of (2.20) we see that

$$(5.7)\qquad \begin{aligned} g(\mathrm{id}, 4m, 4D) \mid T(p^2) &= g(\mathrm{id}, 4m, 4D) \qquad (p \mid 2m),\\ g(\mathrm{id}, 4m, 4D) \mid T(p^2) &= pg(\mathrm{id}, 4m, 4D) \qquad (p \mid Dm^{-1}). \end{aligned}$$

From Lemma 4.8 we obtain

$$\begin{aligned} V(g(\mathrm{id}, 4m, 4D), 1) &= -\prod_{p \mid D/m} p(1+p)^{-1} \sum_{d \mid D/m} \mu(d) 4^{-1}(1+i)\mu(md)(md)^{-1}\\ &= -4^{-1}(1+i)\mu(m)m^{-1}. \end{aligned}$$

Using the equality $g \mid T(4) = g$ we can show, as we did in the proof of Lemma 4.8, that $V(g(\mathrm{id}, 4m, 4D), 1/2\beta) = 0$ for any β, and

$$V(g(\mathrm{id}, 4m, 4D), 1/4) = -4(1+i)^{-1} V(g(\mathrm{id}, 4m, 4D), 1) = \mu(m)m^{-1}.$$

Hence our assertion for $m \neq D$ in the case $l = 1$ follows from Lemma 4.6.

Now consider the case $l \neq 1$. Since

$$g(\chi_l, 4m, 4D) = g(\mathrm{id}, 4m, 4D) | T(l),$$

$g(\chi_l, 4m, 4D)$ belongs to $\mathfrak{S}(4D, \chi_l)$ (see [**6**, Proposition 1.5]), and

$$\begin{aligned} V(g(\chi_l, 4m, 4D), 1) &= l^{-1} \sum_{d \mid l} d^{3/2} \sum_{\substack{k=1\\(k, l/d)=1}}^{l/d} V(g(\mathrm{id}, 4m, 4D), k/ld^{-1})\\ &= l^{-1} \sum_{d \mid l} d^{3/2} \sum_{k=1}^{l/d} \left(\frac{k}{l/d}\right) V(g(\mathrm{id}, 4m, 4D), 1/ld^{-1})\\ &= -4^{-1}(1+i)\mu(m)m^{-1}l^{1/2}. \end{aligned}$$

The form $g(\chi_l, 4m, 4D)$ also satisfies (5.7); hence the case $l \neq 1$ can be proved as in the case $l = 1$.

LEMMA 5.2. *The form $g(\chi_l, m, 4D)$ $(m \neq 1)$ belongs to $\mathfrak{S}(4D, \chi_l)$ and*

$$V(g(\chi_l, m, 4D), 1/\alpha) = -4^{-1}(1+i)\mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$

It vanishes at the other cusps in $S(4D)$.

PROOF. Suppose we have already proved that $g(\chi_l, m, 4D)$ belongs to $\mathfrak{S}(4D, \chi_l)$; then we see that $g(\chi_l, m, 4D)$ satisfies

$$(5.8)\qquad \begin{aligned} g(\chi_l, m, 4D) \mid T(p^2) &= g(\chi_l, m, 4D) \qquad (p \mid m),\\ g(\chi_l, m, 4D) \mid T(p^2) &= pg(\chi_l, m, 4D) \qquad (p \mid 2Dm^{-1}). \end{aligned}$$

From the equality $g \mid T(4) = 2g$ we find

$$\begin{aligned} 2V(g(\chi_l, m, 4D), 1/4\beta) &= 4^{-1} \sum_{k=1}^{4} V(g(\chi_l, m, 4D), (1+4\beta k)/16\beta)\\ &= V(g(\chi_l, m, 4D), 1/4\beta); \end{aligned}$$

hence $V(g(\chi_l, m, 4D), 1/4\beta) = 0$ for any β. Furthermore we have

$$2V(g(\chi_l, m, 4D), 1/2\beta) = 4^{-1} \sum_{k=1}^{4} V(g(\chi_l, m, 4D), (1 + 2\beta k)/8\beta) = 0.$$

For the same reason as in Lemma 5.1, it is sufficient to prove this lemma in the case $l = 1$. Put

$$f_3(\mathrm{id}, 4D) = 2\pi \sum_{n=1}^{\infty} \lambda(n, 4D)\Big(\prod_{p|D} A(p, n) - D^{-1}\Big) n^{1/2} e(nz).$$

Then, by (2.21),

$$\begin{aligned} f_1(\mathrm{id}, 4D) &= 1 - 4\pi(1 + i) \sum_{n=1}^{\infty} \lambda(n, 4D)(A(2, n) - 4^{-1}(1 - i)) \\ &\quad \times \prod_{p|D} A(p, n) n^{1/2} e(nz) - f_3(\mathrm{id}, 4D) \\ &= D^{-1} \sum_{m|D} mg(\mathrm{id}, 4m, 4D) - f_3(\mathrm{id}, 4D). \end{aligned}$$

Therefore, $f_3(\mathrm{id}, 4D)$ belongs to $\mathfrak{S}(4D, \mathrm{id})$ and

$$\begin{aligned} V(f_3(\mathrm{id}, 4D), 1) &= D^{-1} \sum_{m|D} mV(g(\mathrm{id}, 4m, 4D), 1) - V(f_1(\mathrm{id}, 4D), 1) \\ &= -4^{-1}(1 + i)D^{-1} \sum_{m|D} \mu(m) + (1 + i)(4D)^{-1} = (1 + i)(4D)^{-1}, \end{aligned}$$

because $m \neq 1$ implies $D \neq 1$.

If $D = p$ is a prime, then $g(\mathrm{id}, p, 4p) = f_3(\mathrm{id}, 4p)$, which proves our first assertion. Therefore the above computation together with Lemma 4.6 proves the last two assertions when D is a prime. Now we use induction on ν. Since

$$\begin{aligned} &\prod_{p|\beta} (1 + p)^{-1} \prod_{p|D} (A(p, n) - p^{-1}) \\ &\quad = \prod_{p|D/\beta} (A(p, n) - p^{-1}) \prod_{p|\beta} \{(1 + A(p, n))(1 + p)^{-1} - p^{-1}\} \\ &\quad = \sum_{d|\beta} \mu(\beta/d) d\beta^{-1} \prod_{p|D/\beta} (A(p, n) - p^{-1}) \prod_{p|d} (1 + A(p, n))(1 + p)^{-1}, \end{aligned}$$

we obtain

$$\begin{aligned} &\sum_{\substack{\beta|D \\ \beta\neq D}} \mu(\beta) \prod_{p|\beta} (1 + p)^{-1} \prod_{p|D} (A(p, n) - p^{-1}) \\ &\quad = \prod_{p|D} A(p, n) - D^{-1} + \sum_{\substack{\beta|D \\ \beta\neq D}} \sum_{\substack{d|\beta \\ d\neq 1}} \mu(d) d\beta^{-1} \prod_{p|D/\beta} (A(p, n) - p^{-1}) \\ &\quad\quad \times \prod_{p|d} (1 + A(p, n))(1 + p)^{-1}. \end{aligned}$$

Therefore

$$\sum_{\substack{\beta|D\\ \beta\neq D}} \mu(\beta)\prod_{p|\beta}(1+p)^{-1}g(\mathrm{id},D,4D)$$

$$= f_3(\mathrm{id},4D) + \sum_{\substack{\beta|D\\ \beta\neq D}}\sum_{\substack{d|\beta\\ d\neq 1}} \mu(d)d\beta^{-1}\prod_{p|d}(1+p)^{-1}g(\mathrm{id},D\beta^{-1},4Dd^{-1}).$$

(Note the calculation which we did in Lemma 2.2.) By the hypothesis of the induction, this shows $g(\mathrm{id},D,4D)$ belongs to $\mathfrak{S}(4D,\mathrm{id})$ and

$$\sum_{\substack{\beta|D\\ \beta\neq D}} \mu(\beta)\prod_{p|\beta}(1+p)^{-1}V(g(\mathrm{id},D,4D),1)$$

$$= (1+i)(4D)^{-1} + \sum_{\substack{\beta|D\\ \beta\neq D}}\sum_{\substack{d|\beta\\ d\neq 1}} \mu(d)d\beta^{-1}\prod_{p|d}(1+p)^{-1}(-4^{-1}(1+i)\mu(D/\beta)\beta D^{-1})$$

$$= -(1+i)(4D)^{-1}\mu(D)\sum_{\substack{\beta|D\\ \beta\neq D}}\mu(\beta)\prod_{p|\beta}(1+p)^{-1};$$

hence $V(g(\mathrm{id},D,4D),1) = -(1+i)(4D)^{-1}\mu(D)$. Similarly, as (5.6), we have

$$\prod_{p|D/m}(1+p)p^{-1}g(\mathrm{id},m,4D) = \sum_{d|D/m}\mu(d)g(\mathrm{id},md,4md).$$

Therefore we find $V(g(\mathrm{id},m,4D),1) = -(1+i)(4m)^{-1}\mu(m)$. By (5.8) and Lemma 4.6, this completes the proof.

To find a basis of $\mathfrak{S}(4D,\chi_l)$, we define

$$G(\chi_l,4,4D) = l^{-1/2}\varepsilon_l^{-1}g(\chi_l,4,4D),$$

and for every prime p with $p \mid D$ we define

$$G(\chi_l,p,4D) = 2(i-1)l^{-1/2}(l,p)^{1/2}\varepsilon_{p/(l,p)}\left(\frac{l/(l,p)}{p/(l,p)}\right)g(\chi_l,p,4D).$$

Furthermore, suppose we have already defined $G(\chi_l,4\alpha,4D)$ for $\alpha \mid m$, $\alpha\neq m$, and $G(\chi_l,\alpha,4D)$ for $\alpha\mid m$, $\alpha\neq 1, m$; then for $m\ (\neq 1)$ we define

$$G(\chi_l,4m,4D) = l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right)$$

$$\times\Bigg\{g(\chi_l,4m,4D) - g(\chi_l,m,4D) - \mu(m)m^{-1}l^{1/2}$$

$$\times\sum_{\substack{\alpha|m\\ \alpha\neq m}}\mu(\alpha)\alpha(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right)G(\chi_l,4\alpha,4D)\Bigg\},$$

$$G(\chi_l, m, 4D) = 2(i-1)l^{-1/2}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right)$$
$$\times\left\{g(\chi_l, m, 4D) + (1+i)(4m)^{-1}\mu(m)\right.$$
$$\left.\times\sum_{\substack{\alpha|m\\ \alpha\neq 1,m}}\mu(\alpha)\alpha l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right)G(\chi_l,\alpha,4D)\right\}.$$

We can see that $G(\chi_l, 2^r m, 4D)$ ($r = 0$ or 2) vanishes at all cusps in $S(4D)$ except 1 and $1/2^r m$. By a straightforward calculation using Lemmas 5.1 and 5.2 we obtain

$$V(G(\chi_l, 4m, 4D), 1/4m) = V(G(\chi_l, m, 4D), 1/m) = 1,$$
$$(5.9)\quad V(G(\chi_l, 4m, 4D), 1) = -(1+i)(4m)^{-1}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right),$$
$$V(G(\chi_l, m, 4D), 1) = -m^{-1}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right).$$

Therefore

$$G(\chi_l, 4m, 4D)\quad (m\,|\,D),\qquad G(\chi_l, m, 4D)\quad (m\,|\,D,\, m\neq 1)$$

form a basis of $\mathfrak{S}(4D, \chi_l)$ because of (5.5). This proves the theorem for $\mathfrak{S}(4D, \chi_l)$. It is easy to see that

$$g(\chi_l, 4m, 4D)\quad (m\,|\,D),\qquad g(\chi_l, m, 4D)\quad (m\,|\,D,\, m\neq 1)$$

also form a basis of $\mathfrak{S}(4D, \chi_l)$.

6. Proof of Theorem for $\mathfrak{S}(8D, \chi_l)$ and $\mathfrak{S}(8D, \chi_{2l})$. We know that

$$(6.1)\qquad \dim\mathfrak{S}(8D, \chi_l) = \dim\mathfrak{S}(8D, \chi_{2l}) = 3\cdot 2^\nu - 1,$$

from (5.2) and (5.4). Put

$$f_4(\mathrm{id}, 8D) = 2\pi\sum_{n\in R}\lambda(n, 4D)\prod_{p|D}\left(A(p,n) - p^{-1}\right)n^{1/2}e(nz),$$

where $R = \{n\,|\,n\in\mathbf{Z},\ n\geqslant 1,\ n\equiv 1 \text{ or } 2 \pmod 4\}$. Then we have, by (2.27) and (2.28),

$$f_2^*(\mathrm{id}, 4D) + 2^{-1}(1+i)\mu(D)f_2(\mathrm{id}, 8D) = 3/2\cdot f_4(\mathrm{id}, 8D).$$

Here we use $A(2,n) - 4^{-1}(1-i) = 3(i-1)/8$ if $n\in R$ (see (2.17)). By means of Lemmas 4.8 and 4.10 we obtain

$$V(f_4(\mathrm{id}, 8D), 1/8\beta) = V(f_4(\mathrm{id}, 8D), 1/2\beta) = 0,$$
$$(6.2)\qquad V(f_4(\mathrm{id}, 8D), 1/\beta) = -8^{-1}(1+i)\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1},$$
$$V(f_4(\mathrm{id}, 8D), 1/4\beta) = \mu(D/\beta)\beta D^{-1}.$$

Now we define

$$g(\chi_l, 4m, 8D) = 2\pi l^{1/2} \sum_{ln \in R} \lambda(ln, 4D) \prod_{p|m} (A(p, ln) - p^{-1}) n^{1/2} e(nz).$$

LEMMA 6.1. *The form $g(\chi_l, 4m, 8D)$ belongs to $\mathfrak{S}(8D, \chi_l)$ and*

$$V(g(\chi_l, 4m, 8D), 1/\alpha) = -8^{-1}(1+i)\mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)} \right),$$

$$V(g(\chi_l, 4m, 8D), 1/4\alpha) = \mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)} \right).$$

It vanishes at the other cusps in $S(8D)$.

PROOF. Since $g(\chi_l, 4m, 8D) = g(\mathrm{id}, 4m, 8D) \mid T(l)$, it is sufficient to prove this lemma for $l = 1$. We have

$$g(\mathrm{id}, 4m, 8D) = \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) f_4(\mathrm{id}, 8md),$$

hence $g(\mathrm{id}, 4m, 8D)$ belongs to $\mathfrak{S}(8D, \mathrm{id})$. Furthermore from (6.2) we can derive

$$V(g(\mathrm{id}, 4m, 8D), 1/8\beta) = V(g(\mathrm{id}, 4m, 8D), 1/2\beta) = 0,$$

$$V(g(\mathrm{id}, 4m, 8D), 1) = -8^{-1}(1+i)\mu(m)m^{-1}.$$

Observe that $g(\mathrm{id}, 4m, 8D) \mid T(4) = 0$. Therefore we have

$$V(g(\mathrm{id}, 4m, 8D), 1/4) = -8(1+i)^{-1} V(g(\mathrm{id}, 4m, 8D), 1) = \mu(m)m^{-1}.$$

We can also find that

$$g(\mathrm{id}, 4m, 8D) \mid T(p^2) = g(\mathrm{id}, 4m, 8D) \qquad (p \mid m),$$

$$g(\mathrm{id}, 4m, 8D) \mid T(p^2) = p g(\mathrm{id}, 4m, 8D) \qquad (p \mid Dm^{-1}).$$

Therefore the lemma can be proved by Lemma 4.6.

Now we give a basis of $\mathfrak{S}(8D, \chi_l)$. Since $1/8\alpha$ is $\Gamma_0(4D)$-equivalent to $1/4\alpha$, Lemmas 5.1 and 4.2 show that

$$V(g(\chi_l, 4m, 4D), 1/8\alpha) = \mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\frac{2\alpha/(l,\alpha)}{l/(l,\alpha)} \right).$$

We define

$$G(\chi_l, 4, 8D) = l^{-1/2} \varepsilon_l^{-1} g(\chi_l, 4, 8D),$$

$$G(\chi_l, 8, 8D) = l^{-1/2} \varepsilon_l^{-1} \chi_2(l) \{ g(\chi_l, 4, 4D) - g(\chi_l, 4, 8D) \}.$$

For every $m \neq 1$ we define

$$G(\chi_l, m, 8D) = G(\chi_l, m, 4D),$$

$$G(\chi_l, 4m, 8D) = l^{-1/2} (l, m)^{1/2} \varepsilon_{l/(l,m)}^{-1} \left(\frac{m/(l,m)}{l/(l,m)} \right)$$

$$\times \Bigg\{ g(\chi_l, 4m, 8D) - 2^{-1} g(\chi_l, m, 4D) - \mu(m) m^{-1} l^{1/2}$$

$$\times \sum_{\alpha|m,\, \alpha \neq m} \mu(\alpha)\alpha (l, \alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)} \right) G(\chi_l, 4\alpha, 8D) \Bigg\},$$

$$G(\chi_l, 8m, 8D) = l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{2m/(l,m)}{l/(l,m)}\right)$$
$$\times\left\{g(\chi_l,4m,4D) - g(\chi_l,4m,8D) - 2^{-1}g(\chi_l,m,4D) - \mu(m)m^{-1}l^{1/2}\right.$$
$$\left.\times\sum_{\alpha|m,\,\alpha\neq m}\mu(\alpha)\alpha(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\frac{2\alpha/(l,\alpha)}{l/(l,\alpha)}\right)G(\chi_l,8\alpha,8D)\right\}.$$

Then by a straightforward calculation using Lemmas 5.1, 5.2 and 6.1, we have

$$V(G(\chi_l,m,8D),1/m) = 1 \qquad (m\neq 1),$$
$$V(G(\chi_l,4m,8D),1/4m) = V(G(\chi_l,8m,8D),1/8m) = 1,$$
$$(6.3)\quad V(G(\chi_l,m,8D),1) = -m^{-1}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right) \qquad (m\neq 1),$$
$$V(G(\chi_l,4m,8D),1) = -8^{-1}(1+i)m^{-1}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right),$$
$$V(G(\chi_l,8m,8D),1) = -8^{-1}(1+i)m^{-1}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{2m/(l,m)}{l/(l,m)}\right).$$

They vanish at the other cusps in $S(8D)$. Therefore we see that $G(\chi_l,m,8D)$ $(m\neq 1, m\,|\,D)$, $G(\chi_l,4m,8D)$ $(m\,|\,D)$, $G(\chi_l,8m,8D)$ $(m\,|\,D)$, form a basis of $\mathfrak{S}(8D,\chi_l)$ in view of (6.1). This proves the theorem for $\mathfrak{S}(8D,\chi_l)$.

Now we consider $\mathfrak{S}(8D,\chi_{2l})$. Define

$$(6.4)\quad \begin{aligned} g(\chi_{2l},m,8D) &= g(\chi_l,m,4D)\,|\,T(2) \qquad (m\neq 1),\\ g(\chi_{2l},2m,8D) &= g(\chi_l,4m,8D)\,|\,T(2),\\ g(\chi_l,8m,8D) &= g(\chi_l,4m,4D)\,|\,T(2). \end{aligned}$$

The following three lemmas can be proved by the same technique as in the proofs of Lemmas 5.1, 5.2 and 6.1.

LEMMA 6.2. *The form $g(\chi_{2l},m,8D)$ $(m\neq 1)$ belongs to $\mathfrak{S}(8D,\chi_{2l})$ and*

$$V(g(\chi_{2l},m,8D),1/\alpha) = -2^{-3/2}(1+i)\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}$$
$$\times\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$

It vanishes at the other cusps in $S(8D)$.

LEMMA 6.3. *The form $g(\chi_{2l},2m,8D)$ belongs to $\mathfrak{S}(8D,\chi_{2l})$ and*

$$V(g(\chi_{2l},2m,8D),1/\alpha)$$
$$= -2^{-5/2}(1+i)\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right),$$
$$V(g(\chi_{2l},2m,8D),1/2\alpha)$$
$$= 2^{-1}(1+i)\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\varepsilon_l^{-1}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$

It vanishes at the other cusps in $S(8D)$.

LEMMA 6.4. *The form* $g(\chi_{2l}, 8m, 8D)$ *belongs to* $\mathfrak{S}(8D, \chi_{2l})$ *and*

$$V(g(\chi_{2l}, 8m, 8D), 1/\alpha) = -2^{-3/2}(1+i)\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right),$$

$$V(g(\chi_{2l}, 8m, 8D), 1/2\alpha) = 2^{-1}(1+i)\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\varepsilon_l^{-1}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right),$$

$$V(g(\chi_{2l}, 8m, 8D), 1/8\alpha) = \mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right).$$

To find a basis of $\mathfrak{S}(8D, \chi_{2l})$, we define

$$G(\chi_{2l}, 2, 8D) = (1-i)l^{-1/2}\varepsilon_l g(\chi_{2l}, 2, 8D),$$

$$G(\chi_{2l}, 8, 8D) = l^{-1/2}\varepsilon_l^{-1}\{g(\chi_{2l}, 8, 8D) - g(\chi_{2l}, 2, 8D)\}.$$

For every prime $p \mid D$, we define

$$G(\chi_{2l}, p, 8D) = 2^{1/2}(i-1)l^{-1/2}(1, p)^{1/2}\varepsilon_{p/(l,p)}\left(\frac{l/(l,p)}{p/(l,p)}\right)g(\chi_{2l}, p, 8D).$$

For every $m \neq 1$, we define

$$G(\chi_{2l}, m, 8D) = 2^{1/2}(i-1)l^{-1/2}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{2l/(l,m)}{m/(l,m)}\right)$$
$$\times\left\{g(\chi_{2l}, m, 8D) + 2^{-3/2}(1+i)\mu(m)m^{-1}\right.$$
$$\left.\times\sum_{\alpha\mid m,\, \alpha\neq 1, m}\mu(\alpha)\alpha l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right)G(\chi_{2l}, \alpha, 8D)\right\},$$

$$G(\chi_{2l}, 2m, 8D) = (1-i)l^{-1/2}(l,m)^{1/2}\varepsilon_{m/(l,m)}\varepsilon_l\left(\frac{l/(l,m)}{m/(l,m)}\right)$$
$$\times\left\{g(\chi_{2l}, 2m, 8D) - 2^{-1}g(\chi_{2l}, m, 8D) - 2^{-1}(1+i)\mu(m)m^{-1}l^{1/2}\varepsilon_l^{-1}\right.$$
$$\left.\times\sum_{\alpha\mid m,\, \alpha\neq m}\mu(\alpha)(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right)G(\chi_{2l}, 2\alpha, 8D)\right\},$$

$$G(\chi_{2l}, 8m, 8D) = l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right)$$
$$\times\left\{g(\chi_{2l}, 8m, 8D) - g(\chi_{2l}, 2m, 8D) - 2^{-1}g(\chi_{2l}, m, 8D) - \mu(m)m^{-1}l^{1/2}\right.$$
$$\left.\times\sum_{\alpha\mid m,\, \alpha\neq m}\mu(\alpha)\alpha(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right)G(\chi_{2l}, 8\alpha, 8D)\right\}.$$

Then by means of above lemmas we have

$$(6.5)\qquad \begin{aligned} V(G(\chi_{2l}, m, 8D), 1/m) &= V(G(\chi_{2l}, 2m, 8D), 1/2m) \\ &= V(G(\chi_{2l}, 8, 8D), 1/8m) = 1, \end{aligned}$$

$$V(G(\chi_{2l}, m, 8D), 1) = -m^{-1}(l, m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l, m)}{m/(l, m)}\right),$$

$$V(G(\chi_{2l}, 2m, 8D), 1) = -2^{-3/2}m^{-1}(l, m)^{1/2}\varepsilon_{m/(l,m)}\varepsilon_l\left(\frac{l/(l, m)}{m/(l, m)}\right),$$

$$V(G(\chi_{2l}, 8m, 8D), 1) = -2^{-5/2}(1+i)m^{-1}(l, m)^{1/2}\varepsilon^{-1}_{l/(l,m)}\left(\frac{m/(l, m)}{l/(l, m)}\right).$$

This shows that $G(\chi_{2l}, m, 8D)$ $(m \neq 1, m \mid D)$, $G(\chi_{2l}, 2m, 8D)$ $(m \mid D)$, and $G(\chi_{2l}, 8m, 8D)$ $(m \mid D)$, form a basis of $\mathfrak{S}(8D, \chi_{2l})$. Now the theorem is proved for $\mathfrak{S}(8D, \chi_{2l})$.

7. Proof of theorem for $\mathfrak{S}(2^e, \mathrm{id})$ $(e \geq 4)$. First of all we state two well-known facts about Dirichlet characters.

LEMMA 7.1. *Let ψ_i $(1 \leq i \leq \phi(n))$ be all the characters modulo n, where n is a positive integer, and a_j $(1 \leq j \leq \phi(n))$ a full set of representatives of $(\mathbf{Z}/n\mathbf{Z})^*$. Then the matrix $(\psi_i(a_j))$ is nonsingular.*

LEMMA 7.2. *Let ω be a primitive character modulo r, and a, n integers with $r \nmid n$. Then $\sum_{b=1}^{r/(n,r)} \omega(a + nb) = 0$.*

We are going to construct a basis of $\mathfrak{S}(2^e, \mathrm{id})$. Let us take positive integers $c = 2^r$ and m such that $c \mid 2^e$, $m \mid (c, 2^e/c)$. Suppose $m \neq 2$ and ψ is a primitive character modulo m.

Case 1. $2^{e-2} \leq c \leq 2^e$, $m = 1$. Note that if $c = 2^{e-1}$ or 2^e, then m is always 1. Define

$$\begin{aligned} G(\mathrm{id}, 2^e) &= G(\chi_{2^{e-3}}, 8, 8) \mid V(2^{e-3}), \\ G(\mathrm{id}, 2^{e-1}) &= G(\chi_{2^{e-4}}, 8, 8) \mid V(2^{e-4}) - G(\chi_{2^{e-3}}, 8, 8) \mid V(2^{e-3}), \\ G(\mathrm{id}, 2^{e-2}) &= G(\chi_{2^{e-2}a}, a, a) \mid V(2^{e-2}/a) - G(\chi_{2^{e-4}}, 8, 8) \mid V(2^{e-4}), \end{aligned}$$

where $a = (8, 2^{e-2})$. The conductor of $\chi_{2^{e-2}a}$ always divides a. These three forms belong to $\mathfrak{S}(2^e, \mathrm{id})$, and

$$(7.1)\qquad \begin{gathered} V(G(\mathrm{id}, 2^e), 1/2^e) = 1, \quad V(G(\mathrm{id}, 2^{e-1}), 1/2^{e-1}) = 1, \\ V(G(\mathrm{id}, 2^{e-1}), 1/2^e) = 0, \quad V(G(\mathrm{id}, 2^{e-2}), d/2^{e-2}) = \left(\frac{a}{d}\right)\varepsilon_d, \\ V(G(\mathrm{id}, 2^{e-2}), 1/2^s) = 0 \qquad (s = e-1, e), \end{gathered}$$

by (6.3), (6.5) and (5.9).

Case 2. $2^5 \leqslant c \leqslant 2^{e-5}$, $\psi^4 \neq \mathrm{id}$. Note that if $\psi^4 \neq \mathrm{id}$, then $m \geqslant 2^5$. Define

$$g = g(\psi, c) = E(\psi^{-2}\chi_c, m) \mid W(m),$$
$$h = h(\psi, c) = \sum_{k=1}^{m} \psi(k) g(z + k/m),$$
$$G = G(\psi, c) = h \mid V(c/m).$$

Then we know that $g \in \mathfrak{S}(m, \psi^2\chi_{cm})$ by Lemma 3.1; $h \in \mathfrak{S}(m^2, \chi_{cm})$ by Lemma 3.2; and $G \in \mathfrak{S}(mc, \mathrm{id}) \subset \mathfrak{S}(2^e, \mathrm{id})$. Since g vanishes at all cusps of $S(m)$ except 1 by Lemma 4.11, and

$$h(z + d/\alpha) = \sum_{k=1}^{m} \psi(k) g(z + d/\alpha + k/m),$$

h does not vanish only at cusps d/m. Furthermore we have $V(h, d/m) = \psi(-d)m^{3/2}V(g, 1)$. Therefore

$$V(G, d/c) = \rho\psi(d), \quad V(G, d/\alpha) = 0 \qquad (c \mid \alpha, c \neq \alpha). \tag{7.2}$$

Here and below we denote by ρ a nonzero quantity which does not depend on d.

Let us put

$$g_1 = G(\mathrm{id}, 4, 8), \quad g_2 = G(\chi_2, 2, 8),$$
$$h_1(\psi)(z) = \sum_{k=1}^{m} \psi(k) g_1(z + k/m),$$
$$h_2(\psi)(z) = \sum_{k=1}^{m} \psi(k) g_2(z + k/m).$$

Case 3. $4 \leqslant c \leqslant 2^{e-3}$ or $c = 2^{e-2}$, $m = 4$; $2 \mid r$; $\psi^4 = \mathrm{id}$. The condition $\psi^4 = \mathrm{id}$ implies $m = 1, 4, 8, 16$.

(i) $m = 1$. Define $G(\mathrm{id}, c) = G(\chi_2, 2, 8) \mid V(c/2)$. It belongs to $\mathfrak{S}(4c, \mathrm{id}) \subset \mathfrak{S}(2^e, \mathrm{id})$, and satisfies (7.2).

(ii) $m = 4$. Define $G(\chi_{-1}, c) = h_1(\chi_{-1}) \mid V(c/4)$. We also have $G(\chi_{-1}, c) \in \mathfrak{S}(2^e, \mathrm{id})$. It is easy to verify that $V(h_1(\chi_{-1}), d/4) = 4^{3/2}\chi_{-1}(-d)V(g_1, 1)$ and $V(h_1(\chi_{-1}), d/\alpha) = 0$ if $\alpha = 8, 16$. Hence $G(\chi_1, c)$ satisfies (7.2).

(iii) $m = 8$ or 16. Define $G(\psi, c) = h_2(\psi) \mid V(c/m)$. It belongs to $\mathfrak{S}(2^e, \mathrm{id})$. The form $h_2(\psi)$ does not vanish only at cusps d/m, and

$$\begin{aligned} V(h_2(\psi), d/m) &= \psi(-d)m^{3/2}V(g_2, 1) + \psi(m/2 - d)(m/2)^{3/2} \\ &= -2^{-1/2}\psi(-d)m^{3/2}. \end{aligned}$$

Here we use $V(g_2, 1) = -2^{-3/2}$ by (6.5). Therefore $G(\psi, c)$ satisfies (7.2).

There exist four characters ω_i $(i \leqslant 4)$ with conductor 16:

$$\omega_1(a) = e^{t\pi i/2}, \quad \omega_2(a) = e^{3t\pi i/2},$$
$$\omega_3(a) = \chi_{-1}(a)\omega_1(a), \quad \omega_4(a) = \chi_{-1}(a)\omega_2(a)$$

if $a \equiv \pm 5^t \pmod{16}$, $1 \leqslant t \leqslant 4$.

Case 4. $8 \leqslant c \leqslant 2^{e-3}$ or $c = 2^{e-2}$, $m = 4$; $2 \nmid r$; $\psi^4 = \mathrm{id}$, $\psi \neq \chi_{-2}, \omega_4$.

(i) $m = 1$. Define $G(\mathrm{id}, c) = G(\mathrm{id}, 8, 8) \mid V(c/8) - G(\mathrm{id}, 4, 4) \mid V(c/2)$. Then $G(\mathrm{id}, c) \in \mathfrak{S}(2^e, \mathrm{id})$ and

$$(7.3)\quad \begin{aligned} V(G(\mathrm{id}, c), d/c) &= V(G(\mathrm{id}, 8, 8), d/8) - V(G(\mathrm{id}, 4, 4), 1/2) \\ &= \chi_2(d)\varepsilon_d = 2^{-1}(1+i)\chi_2(d) + 2^{-1}(1-i)\chi_{-2}(d), \\ V(G(\mathrm{id}, c), d/\alpha) &= 0 \qquad (c \mid \alpha, c \neq \alpha). \end{aligned}$$

(ii) $m = 4$. If $e \leqslant 2^{e-3}$, define $G(\chi_1, c) = h_2(\chi_{-1}) \mid V(c/4)$. It belongs to $\mathfrak{S}(2^e, \mathrm{id})$. We have

$$V(h_2(\chi_{-1}), d/4) = 8\chi_{-1}(-d)V(g_2, 1) + 2^{3/2}\chi_{-1}(2-d) = 2^{5/2}\chi_{-1}(d)$$

and $V(h_2(\chi_{-1}), d/\alpha) = 0$ $(\alpha = 8, 16)$. Hence $G(\chi_{-1}, c)$ satisfies (7.2). If $c = 2^{e-2}$, define $h(z) = h_2(\chi_{-1})(-1/32z)z^{-3/2}$, $G(\chi_{-1}, c) = h \mid V(c/8)$. It is easy to see that $h \in \mathfrak{S}(32, \mathrm{id})$ and $G(\chi_{-1}, c) \in \mathfrak{S}(2^e, \mathrm{id})$. By a calculation, we can find

$$V(h, d/8) = 2^{9/2}\left(\frac{2}{d}\right)\varepsilon_d V(h_2(\chi_{-1}), -d/4) = -2^7\left(\frac{2}{d}\right)\varepsilon_d\chi_{-1}(d)$$

and $V(h, d/\alpha) = 0$ $(\alpha \mid 32, \alpha \neq 8)$. Furthermore we have

$$V(G(\chi_{-1}, c), d/c) = -2^7\left(\frac{c}{d}\right)\varepsilon_d\chi_{-1}(d), \quad V(G(\chi_{-1}, c), d/\alpha) = 0 \qquad (\alpha \mid 2^e, \alpha \neq c).$$

(iii) $m = 8$, $\psi = \chi_2$. Define $G(\chi_2, c) = h_1(\chi_2) \mid V(c/8)$. It belongs to $\mathfrak{S}(2^e, \mathrm{id})$. We have

$$\begin{aligned} V(h_1(\chi_2), d/8) &= 8^{3/2}\chi_2(d)V(g_1, 1) \\ &\quad + 2^{3/2}\chi_2(2-d) + 2^{3/2}\chi_2(6-d)V(g_1, 3/4) \\ &= -2^{3/2}\{(1+i)\chi_2(d) - (1-i)\chi_{-2}(d)\}, \\ V(h_1(\chi_2), d/\alpha) &= 0 \qquad (\alpha \neq 8). \end{aligned}$$

Thus

$$(7.4)\quad \begin{aligned} V(G(\chi_2, c), d/c) &= -2^{3/2}\{(1+i)\chi_2(d) - (1-i)\chi_{-2}(d)\}, \\ V(G(\chi_2, c), d/\alpha) &= 0 \qquad (c \mid \alpha, c \neq \alpha). \end{aligned}$$

(iv) $m = 16$, $\psi = \omega_j$ $(j = 1, 2, 3)$. Define

$$h_j = \sum_{k=1}^{m} \omega_j(k)G(\mathrm{id}, 8, 8)(z + (k/16)), \qquad G_j = h_j \mid V(c/16).$$

Since $\omega_j^2 = \chi_2$, $2 \nmid \gamma$, we see that $G_j \in \mathfrak{S}(2^e, \mathrm{id})$. We have

$$(7.5)\quad V(h_j, d/16) = -8(1+i)\omega_j(-d) + 2^{5/2}\big(\omega_j(2-d) - i\omega_j(6-d)\big).$$

Here we use $V(G(\mathrm{id}, 8, 8), 1) = -8^{-1}(1+i)$ by (6.3). If $s \geqslant 5$, then

$$\begin{aligned} V(h_j, d/2^s) &= \sum_{k=1}^{16} \omega_j(k)\varepsilon_{d+2^{s-4}k}\chi_{2^s}(d + 2^{s-4}k) \\ &= \sum_{k=1}^{4} \varepsilon_{d+2^{s-4}k}\chi_{2^s}(d + 2^{s-4}k) \sum_{n=1}^{4} \omega_j(k+4n) = 0 \end{aligned}$$

by Lemma 7.2. After a straightforward calculation we find

$$
\begin{aligned}
&V(G_1, d/c) = -8(1+i)\omega_1(d) + 2^{5/2}(\omega_2(d) + \omega_4(d)),\\
&V(G_2, d/c) = -8(1+i)\omega_2(d) + 2^{5/2}i(\omega_1(d) - \omega_3(d)),\\
&V(G_3, d/c) = 8(1+i)\omega_3(d) + 2^{5/2}(\omega_2(d) + \omega_4(d)),\\
&V(G_i, d/\alpha) = 0 \qquad (i = 1,2,3;\ c \mid \alpha, c \neq \alpha).
\end{aligned}
\tag{7.6}
$$

Let $d_1, d_2, \dots, d_8$ be a full set of representatives of $(\mathbf{Z}/16\mathbf{Z})^*$. Using the fact that $\{\omega_j(d_1), \dots, \omega_j(d_8)\}$ $(j = 1,2,3,4)$ are linearly independent, we can show that $(V(G_j, d_1/c), \dots, V(G_j, d_8/c))$ $(j = 1,2,3)$ are linearly independent. (Note that if we take $1 \leqslant j \leqslant 4$ instead of $1 \leqslant j \leqslant 3$, the four vectors would be linearly dependent.)

Case 5. c and ψ satisfy any one of the following three conditions.

(i) $c = 1, 2$.

(ii) If $e \geqslant 6$, $c = 8$, or $c = 2^{e-3}$, $2 \nmid e-3$; $\psi = \chi_{-2}$.

(iii) If $e \geqslant 9$, $2^5 \leqslant c \leqslant 2^{e-4}$, $2 \nmid \gamma$; $\psi = \chi_{-2}$ or ω_4.

In this case we do not define any more forms.

The total number of pairs (ψ, c), where $c \mid 2^e$, the conductor m of ψ divides $(c, 2^e/c)$, is $\Sigma_{c|2^e} \phi((c, 2^e/c))$. The number of pairs (ψ, c) satisfying (i), or (ii), or (iii) of Case 5 is

$$
\begin{cases}
e-4 & (e \geqslant 7),\\
3 & (e = 6),\\
2 & (e = 5),\\
2 & (e = 4).
\end{cases}
$$

This is exactly the dimension of $\mathfrak{S}(2^e, 1/2, \mathrm{id})$ (see §5). Hence the number of forms $\{G(\psi, c)\}$ which we defined in Cases 1–4 equals the dimension of $\mathfrak{S}(2^e, \mathrm{id})$. Consider the matrix $A = (V(G(\psi, c), s))$ where s is a cusp of $S(2^e)$. Every row of A corresponds to a form $G(\psi, c)$ while every column of A corresponds to a cusp in $S(2^e)$. These rows (columns) are arranged according to the order of c defined in §5. Among those rows (columns) corresponding to the same c, we can arrange them in any order. Thus, by means of Lemma 7.1 and (7.1)–(7.5), we see that the rows of A are linearly independent. This means that $\{G(\psi, c)\}$ form a basis of $\mathfrak{S}(2^e, \mathrm{id})$. Therefore the theorem is proved for $\mathfrak{S}(2^e, \mathrm{id})$.

8. Euler products of some Dirichlet series. Let $f(z) = \Sigma_{n=0}^{\infty} a(n)e(nz)$ be an element of $M(N, \omega)$, that is a common eigenfunction of $T(p^2)$ for all primes p, and let $f \mid T(p^2) = \lambda_p f$. Further let t be a positive square-free integer, prime to N. Then the formal Dirichlet series $\Sigma_{n=1}^{\infty} a(tn^2)n^{-s}$ has the Euler product

$$
\sum_{n=1} a(tn^2)n^{-s} = a(t)\prod_p \left[1 - \omega(p)\left(\frac{-t}{p}\right)p^{-s}\right]\left[1 - \lambda_p p^{-s} + \omega(p^2)p^{1-2s}\right]^{-1}
$$

[**6**, Theorem 1.9].

Now we consider the form $g(\chi_l, 4m, 4D)$. We know that

$$
\begin{aligned}
&g(\chi_l, 4m, 4D) \mid T(p^2) = g(\chi_l, 4m, 4D) \qquad (p \mid 2m),\\
&g(\chi_l, 4m, 4D) \mid T(p^2) = p \cdot g(\chi_l, 4m, 4D) \qquad (p \mid Dm^{-1}).
\end{aligned}
$$

Put

$$a(n)=\lambda(ln,4D)\left[A(2,ln)-4^{-1}(1-i)\right]\prod_{p|m}\left[A(p,ln)-p^{-1}\right]n^{1/2},$$

where $\lambda(ln,4D)$ is defined in (2.26). Suppose q is a prime with $q\nmid 2D$ and $g(\chi_l,4m,4D)\mid T(q^2)=\sum_{n=0}^{\infty}b(n)e(nz)$. It is easy to verify that

$$L_{4D}(1,\chi_{-lnq^2})\left[A(2,lnq^2)-4^{-1}(1-i)\right]\prod_{p|m}\left[A(p,lnq^2)-p^{-1}\right]$$
$$=L_{4D}(1,\chi_{-ln})\left[A(2,ln)-4^{-1}(1-i)\right]\prod_{p|m}\left[A(p,ln)-p^{-1}\right].$$

Let $ln=\tau\sigma^2$, where τ is a positive square-free integer and σ is a positive integer, and $h(p)$ the highest exponent such that $p^{h(p)}\mid ln$. From the expression

$$\beta(\tau\sigma^2,0,\chi_D,4D)$$
$$=\prod_{p|\tau,p\nmid 2D}\left(\sum_{k=0}^{(h(p)-1)/2}p^{-k}\right)\prod_{p|\sigma,p\nmid 2D\tau}\left(\sum_{k=0}^{h(p)/2}p^{-k}-\left(\frac{-l\tau}{p}\right)\sum_{k=1}^{h(p)/2}p^{-k}\right)$$

we find that if $q\mid\tau$, then

$$\beta(\tau\sigma^2q^2,0,\chi_D,4D)=\left(\sum_{k=0}^{(h(q)+1)/2}q^{-k}\right)\left(\sum_{k=0}^{(h(q)-1)/2}q^{-k}\right)^{-1}\beta(\tau\sigma^2,0,\chi_D,4D);$$

if $h(q)=0$, then

$$\beta(\tau\sigma^2q^2,0,\chi_D,4D)=\left(1+q^{-1}-\left(\frac{-l\tau}{q}\right)q^{-1}\right)\beta(\tau\sigma^2,0,\chi_D,4D);$$

if $q\mid\sigma$, $q\nmid\tau$, then

$$\beta(\tau\sigma^2q^2,0,\chi_D,4D)=\left(\sum_{k=0}^{(h(q)/2)+1}q^{-k}-\left(\frac{-\tau l}{q}\right)\sum_{k=1}^{(h(q)/2)+1}q^{-k}\right)$$
$$\times\left(\sum_{k=0}^{h(q)/2}q^{-k}-\left(\frac{-\tau l}{q}\right)\sum_{k=1}^{h(q)/2}q^{-k}\right)^{-1}\beta(\tau\sigma^2,0,\chi_D,4D).$$

From (3.1), if $h(q)=1$, we obtain

$$b(n)=a(q^2n)=(1+q)a(n);$$

if $q\mid\tau$, $h(q)\geqslant 3$, we obtain

$$b(n)=\left\{q\left(\sum_{k=0}^{(h(q)+1)/2}q^{-k}\right)\left(\sum_{k=0}^{(h(q)-1)/2}q^{-k}\right)^{-1}\right.$$
$$\left.+\left(\sum_{k=0}^{(h(q)-3)/2}q^{-k}\right)\left(\sum_{k=0}^{(h(q)-1)/2}q^{-k}\right)^{-1}\right\}a(n)$$
$$=(1+q)a(n);$$

if $h(q)=0$, we obtain

$$b(n)=\left\{q\left(1+q^{-1}-\left(\frac{-l\tau}{q}\right)q^{-1}\right)+\left(\frac{-l\tau}{q}\right)\right\}a(n)=1(1+q)a(n);$$

and if $q\mid\sigma$, $q\nmid\tau$, we obtain

$$\begin{aligned}b(n)=\Bigg\{q&\left(\sum_{k=0}^{(h(q)/2)+1}q^{-k}-\left(\frac{-l\tau}{q}\right)\sum_{k=1}^{(h(q)/2)+1}q^{-k}\right)\\&\times\left(\sum_{k=0}^{h(q)/2}q^{-k}-\left(\frac{-l\tau}{q}\right)\sum_{k=1}^{h(q)/2}q^{-k}\right)^{-1}\\&+\left(\sum_{k=0}^{(h(q)/2)-1}q^{-k}-\left(\frac{-l\tau}{q}\right)\sum_{k=1}^{(h(q)/2)-1}q^{-k}\right)\\&\times\left(\sum_{k=0}^{h(q)/2}q^{-k}-\left(\frac{-l\tau}{q}\right)\sum_{k=1}^{h(q)/2}q^{-k}\right)^{-1}\Bigg\}a(n)\\=(1+q)a(n).&\end{aligned}$$

Therefore we have

$$(*)\qquad g(\chi_l,4m,4D)\mid T(q^2)=(1+q)g(\chi_l,4m,4D)$$

for any prime $q\nmid 4D$. Thus $\sum_{n=1}^{\infty}a(tn^2)n^{-s}$ has the Euler product

$$\begin{aligned}\sum_{n=1}a(tn^2)n^{-s}&=a(t)\prod_{p\mid 2m}(1-p^{-s})^{-1}\prod_{p\mid D/m}(1-p^{1-s})^{-1}\\&\quad\times\prod_{q\nmid 2D}\left(1-\left(\frac{-lt}{q}\right)q^{-s}\right)(1-(1+q)q^{-s}+q^{1-2s})^{-1}\\&=a(t)L_{D/m}(s,\mathrm{id})L_{2m}(s-1,\mathrm{id})L_{2Dt}(s,\chi_{-lt})^{-1},\end{aligned}$$

where t is a positive square-free integer, prime to $2D$.

If we consider $g(\chi_l,m,4D)$, $g(\chi_l,4m,8D)$, $g(\chi_{2l},m,4D)$, $g(\chi_{2l},4m,4D)$ and $g(\chi_{2l},4m,8D)$, they also satisfy $(*)$. Therefore we can obtain corresponding Euler products in the same way.

Now we consider $f_2^*(\omega,N)$. Let θ be a primitive character modulo a positive integer r. From Proposition 2 of [7], we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\beta(tm^2,-1,\omega\chi_N,N)\theta(m)m^{-1-s}\\&\qquad=L_{Nr}\left(s+1,\theta(\omega\chi_N)^{(t)}\right)^{-1}L_{Nr}(s,\theta\omega')L(s+1,\theta).\end{aligned}$$

References

1. H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, Modular Functions of One Variable.VI, Lecture Notes in Math., vol. 627, Springer-Verlag, Berlin and New York, 1977.

2. E. Hecke, *Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 199–224.

3. H. Maass, *Konstruktion ganzer Modulformen halbzahliger Dimension mit I-Multiplikatoren in einer und zwei Variablen*, Abh. Math. Sem. Univ. Hamburg **12** (1937), 133–162.

4. H. Petersson, *Über die Entwicklungskoeftizienten der ganzen Modulformen und ihre Bedentung für die Zahlentheorie*, Abh. Math. Sem. Univ. Hamburg **8** (1931), 215–242.

5. J. P. Serre and H. M. Stark, *Modular forms of weight* 1/2, Modular Functions of One Variable.VI, Lecture Notes in Math., vol. 627, Springer-Verlag, Berlin and New York.

6. G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481.

7. ______, *On the holomorphy of certain Dirichlet series*, Proc. London. Math. Soc. (3) **31** (1975), 79–98.

8. ______, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Tokyo and Princeton Univ. Press, Princeton, N. J., 1971.

9. J. Sturm, *Special values of zeta functions, and Eisenstein series of half integral weight*, Amer. J. Math. **102**, No. 2 (1980), 219–240.

The Insititute of Applied Mathematics, Chinese Academia Sinica, Peking, The People's Republic of China